Optimal discrimination of Gaussian states and channels

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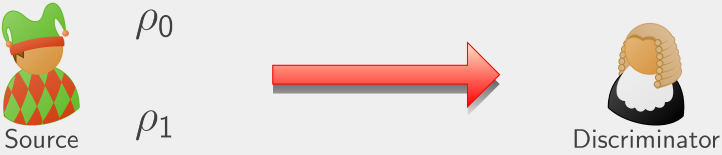
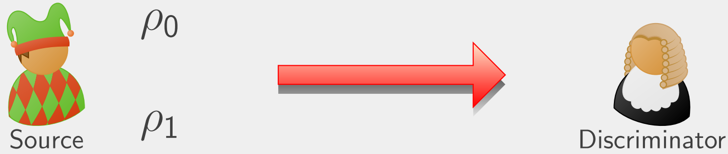
Discrimination of quantum states



- The source generates either ρ_0 or ρ_1 and send it to the discriminator
- The discriminator has to decide which state he has received

Discrimination of quantum states



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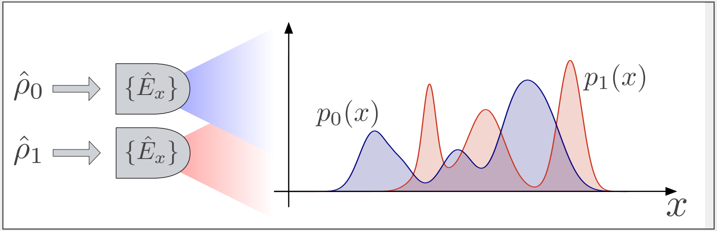
- The discriminator has to decide which state he has received

In this talk ρ_0 and ρ_1 are known!

- $[\rho_0, \rho_1] \neq 0$
- ρ_i may be mixed (e.g. for uncertainty, thermal noise etc.)

Optimal discrimination of quantum states

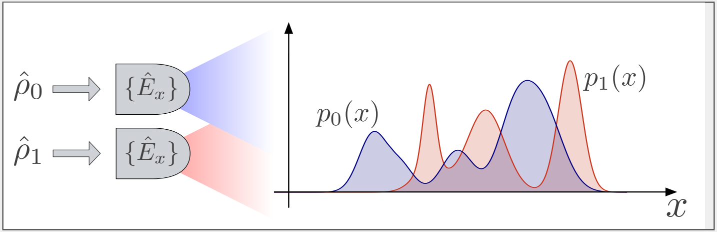


Discriminator strategy: measurement via POVM E_x

$$p_0(x) = \mathrm{Tr}[E_x \rho_0]$$
$$p_1(x) = \mathrm{Tr}[E_x \rho_1]$$

What is the best POVM to distinguish ρ_0 from ρ_1?

Strategy



- Define a distance between the probability distributions

$$D(p_0(x), p_1(x))$$

- Find the optimal POVM to **maximize** that distance

$$E_x = \underset{E_x}{\operatorname{argmax}}\, D(p_0(x), p_1(x))$$

Helstrom measurement

Example:

$$D(p_0(x), p_1(x)) = \frac{1}{2} \int dx \, |p_0(x) - p_1(x)|$$

we get the trace distance

$$\max_{E_x} D(p_0(x), p_1(x)) = D_{\mathrm{Tr}}(\rho_0, \rho_1) := \frac{1}{2} \|\rho_0 - \rho_1\|_1$$

Optimal measurement: Helstrom measurement

$$E_x = |x\rangle\langle x| := \mathrm{eigenprojectors}(\rho_0 - \rho_1)$$

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Holevo-Helstrom bound

If the source sends ρ_0 or ρ_1 with equal probability

$$p_{\mathrm{error}} \geq \frac{1}{2} \left(1 - D_{\mathrm{Tr}}(\rho_0, \rho_1)\right)$$

Relative entropy

Example:

$$D(p_0(x)|p_1(x)) = \frac{1}{2}\int dx\; p_0(x)\log\frac{p_0(x)}{p_1(x)}$$

The optimum is **not** the quantum relative entropy

$$\max_{E_x} D(p_0(x)|p_1(x)) \neq S(\rho_0|\rho_1) := \mathrm{Tr}[\rho_0(\log\rho_0 - \log\rho_1)]$$

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Relative entropy can only be achieved asymptotically

$$D_n(\rho_0|\rho_1) = \sup_{E_x}\frac{1}{n}\int dx\ \mathrm{Tr}\left[E_x\rho_0^{\otimes n}\right]\log\frac{\mathrm{Tr}\left[E_x\rho_0^{\otimes n}\right]}{\mathrm{Tr}\left[E_x\rho_1^{\otimes n}\right]}$$

tight inequality for finite d (Hayashi, '97)

$$D_n(\rho_0|\rho_1) \leq S(\rho_0|\rho_1) \leq D_n(\rho_0,\rho_1) + \frac{d-1}{n}\log(n+1)$$

Relative entropy

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Fidelity

Bhattacharyya distance

$$D(p_0(x), p_1(x)) = \cos^{-1} \int dx \, \sqrt{p_0(x) p_1(x)}$$

The optimum is the Fidelity-based Bures distance

$$\max_{E_x} D(p_0(x), p_1(x)) = \cos^{-1} F(\rho_0, \rho_1)$$

$$F(\rho_0, \rho_1) = \mathrm{Tr} \sqrt{\sqrt{\rho_0} \rho_1 \sqrt{\rho_0}}$$

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Optimal measurement (Fuchs,Caves,'95)

$$E_x = |x\rangle\langle x| = \mathrm{eigenprojections}(M)$$

$$M = \rho_0^{-1/2} \sqrt{\rho_0^{1/2} \rho_1 \rho_0^{1/2}} \rho_0^{-1/2}$$

Fidelity of Gaussian States

Let ρ_j be a Gaussian state with

$$\text{vectot of first moments } \rightarrow u_j$$
$$\text{covariance matrix } \rightarrow V_j$$

Banchi, Braunstein, Pirandola — PRL 2015

$$F(\rho_0, \rho_1) = \frac{\exp\left(-\delta(V_0 + V_1)^{-1}\delta_u/4\right)}{\det^{1/4}(V_0 + V_1)} \times$$
$$\times \det\left[2\left(\sqrt{1 + (2V_{\mathrm{aux}}\Omega)^{-2}} + 1\right) V_{\mathrm{aux}}\right]^{1/4}$$

where $\delta = u_0 - u_1$ and

$$V_{\mathrm{aux}} = \Omega^T (V_0 + V_1)^{-1} \left(\frac{\Omega}{4} + V_1 \Omega V_0\right)$$

Main ingredients

Exponential representation ("hat" means operator):

$$\hat{\rho} = \frac{\exp\left[-(\hat{Q} - u)^T G (\hat{Q} - u)\right]}{\det(V + i\Omega/2)^{1/2}}$$

for "Gibbs matrix"

$$G = 2i\Omega \cosh^{-1}(2Vi\Omega)$$

Banchi, Braunstein, Pirandola, PRL 2015

Matrix functions

Matrix function $F(M)$ is a matrix

$$F(M)_{ij} = \sum_k U_{ik} F(\lambda_k) (U^\dagger)_{kj}$$

where $M = U \lambda U^\dagger$. Very easy numerically.

Analytically it is very easy using Mathematica

In[1]:= $\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$ $\quad V = \begin{pmatrix} \mu & \sqrt{\eta}\sqrt{\mu^2 - \frac{1}{4}} & 0 & 0 \\ \sqrt{\eta}\sqrt{\mu^2 - \frac{1}{4}} & \mu + (1-\eta)/2 & 0 & 0 \\ 0 & 0 & \mu & -\sqrt{\eta}\sqrt{\mu^2 - \frac{1}{4}} \\ 0 & 0 & -\sqrt{\eta}\sqrt{\mu^2 - \frac{1}{4}} & \mu + (1-\eta)/2 \end{pmatrix};$

```
G = 2 I Ω.MatrixFunction[ArcCosh, 2 I V.Ω];
```

Advantage

- It is possible to easily express arbitrary powers

$$\hat{\rho}^{\alpha} \propto e^{-\alpha(\hat{Q}-u)^T G(\hat{Q}-u)}$$

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- Operators logarithms are straightforward

$$\log \hat{\rho} = -(\hat{Q} - u)^T G(\hat{Q} - u) + \ldots$$

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Pirandola, Laurenza, Ottaviani, Banchi, Nat. Comm. 2017

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Pirandola, Laurenza, Ottaviani, Banchi, Nat. Comm. 2017

- Product of states via generalized BCH, e.g.

$$e^{-\alpha\hat{Q}^T G_0 \hat{Q}} e^{-\beta\hat{Q}^T G_1 \hat{Q}} = e^{-\hat{Q} G_{\mathrm{new}} \hat{Q}}$$
$$e^{-\alpha i\Omega G_0} e^{-\beta i\Omega G_1} = e^{-i\Omega G_{\mathrm{new}}}$$

Main ingredients

Combining all these ingredients we found closed form expressions for

$$F(\rho_0, \rho_1) := \mathrm{Tr}\sqrt{\sqrt{\rho_0}\rho_1\sqrt{\rho_0}}$$

$$S(\rho_0|\rho_1) := \mathrm{Tr}[\rho_0(\log\rho_0 - \log\rho_1)]$$

Advantages:

- Calculations via matrix algebra
- No symplectic decomposition needed (just `MatrixFunction`)

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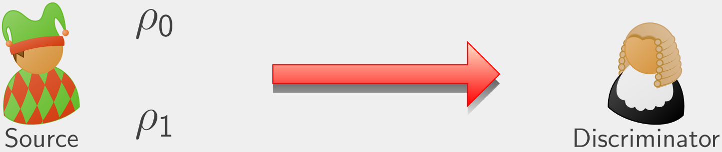
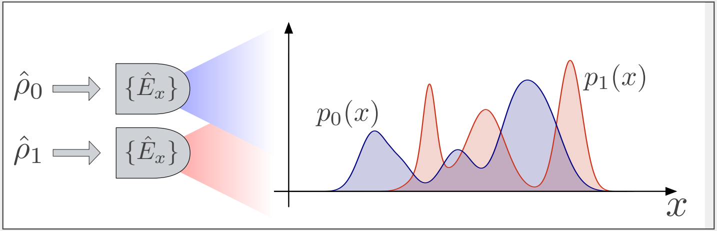
Quantum Fisher Information from $F(\rho, \rho + \delta\rho)$

$$\frac{du^T V^{-1} du}{4} + \frac{\mathrm{Tr}\left[dV(4\mathcal{L}_V + \mathcal{L}_\Omega)^{-1} dV\right]}{2}$$

where $\mathcal{L}_X[Y] = XYX$.

Optimal measurement



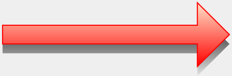


Optimal measurement

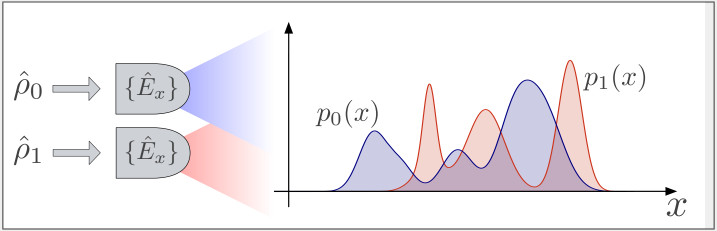
Source

ρ_0

ρ_1



Discriminator



$$E_x = |x\rangle\langle x| = \text{eigenprojections}(M)$$

$$M = \rho_0^{-1/2}\sqrt{\rho_0^{1/2}\rho_1\rho_0^{1/2}}\,\rho_0^{-1/2}$$

Optimal measurement for Gaussian states

$$M = \rho_0^{-1/2}\sqrt{\rho_0^{1/2}\rho_1\rho_0^{1/2}}\rho_0^{-1/2}$$
$$\propto \hat{D}(u_1)\exp\left[-\frac{1}{2}\hat{Q}^{\mathsf{T}} G_{\mathsf{M}}\hat{Q} - v_{\mathsf{M}}^{\mathsf{T}}\hat{Q}\right]\hat{D}^{\dagger}(u_1),$$

where $\hat{D}$ is a displacement operator, v_M depends on first and second moments, and

$$e^{i\Omega G_{\mathsf{M}}} = e^{-i\Omega G_1/2}\sqrt{e^{i\Omega G_1/2}e^{i\Omega G_0}e^{i\Omega G_1/2}}e^{-i\Omega G_1/2}$$

Oh, Lee, Banchi, Lee, Rockstuhl, Jeong, PRA 2019

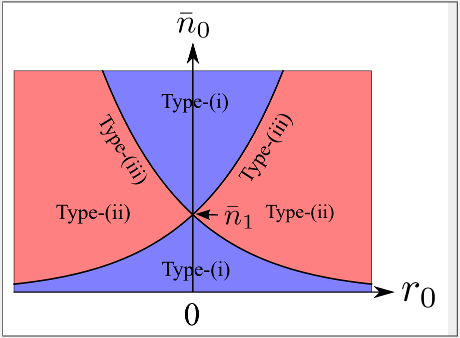
Example: single-mode case

A single-mode state can be decomposed as

$$\hat{\rho} = \hat{D}(u)\hat{S}(\xi)\hat{\rho}_{\mathsf{T}}\hat{S}^{\dagger}(\xi)\hat{D}^{\dagger}(u) = \sum_{n=0}^{\infty} \frac{\bar{n}^{n}}{(\bar{n}+1)^{n+1}}\ \hat{D}(u)\hat{S}(\xi)|n\rangle\langle n|\hat{S}^{\dagger}(\xi)\hat{D}^{\dagger}(u),$$

where $\xi = r^{\theta}$ is the squeezing parameter and $\bar{n}$ the average number of photons in the thermal state $\hat{\rho}_T$.

Classification of optimal measurement for fixed $\bar{n}_1$ as a function of $\bar{n}_0$ and r_0



Full classification of the single mode case

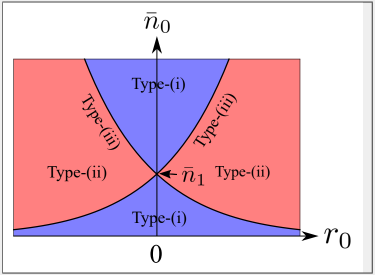
There exists three types of optimal measurements, depending on G_M and v_M

(i) Excitation-number-resolving detection: the number operator $\hat{n} = (\hat{x}^2 + \hat{p}^2 - 1)/2$ followed by the unitary operation $\hat{V}$ and a squeezing operation

(ii) Projection onto the eigenbasis of $\hat{x}\hat{p} + \hat{p}\hat{x}$

(iii) Homodyne detection

Classification of optimal measurement for fixed $\bar{n}_1$ as a function of $\bar{n}_0$ and r_0



Applications in metrology

$$H(\theta) = \frac{4[1 - F(\hat{\rho}_\theta, \hat{\rho}_{\theta+d\theta})]}{d\theta^2} = \mathrm{Tr}[\hat{\rho}_\theta \hat{L}_\theta^2],$$

The SLD $\hat{L}_\theta$ is the optimal measurement to distinguish two infinitesimally close states. The optimal measurement corresponds to the limit

$$\hat{M}(\hat{\rho}_\theta, \hat{\rho}_{\theta+d\theta}) \simeq 1 + \hat{L}_\theta d\theta/2$$

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For a general Gaussian state we find thus the optimal measurement for parameter estimation:

$$\hat{L}_\theta = -\hat{D}(u_\theta)(\hat{Q}^\mathsf{T} G_\mathsf{M} \hat{Q} - 2 v_\mathsf{M}^\mathsf{T} \hat{Q}) \hat{D}^\dagger(u_\theta)$$

with $v_\mathsf{M} = V_\theta^{-1}(\partial u_\theta/\partial\theta) d\theta/2$ and G_M is the solution of

$$4 V_\theta G_\mathsf{M} V_\theta + \Omega G_\mathsf{M} \Omega + 2 \frac{\partial V_\theta}{\partial \theta} = 0$$

Oh, Lee, Banchi, Lee, Rockstuhl, Jeong, PRA 2019

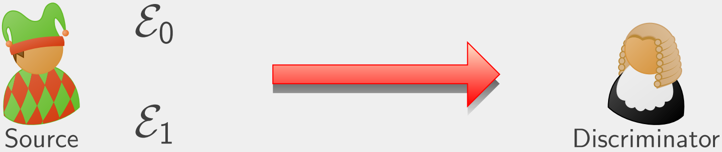
Example: single-mode case

Best measurement for the estimation of

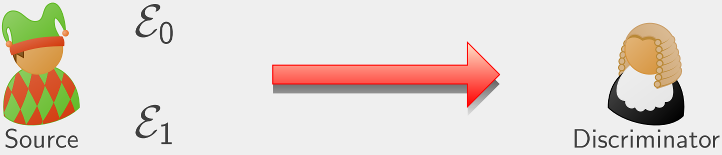
- Displacement: homodyne detection
- Phase or Squeezing: Projection onto the eigenbasis of $\hat{x}\hat{p} + \hat{p}\hat{x}$
  (unknown experimental procedure)
- Loss: excitation-number-resolving detection

The formalism can be applied to any number of modes and for any type of Gaussian states

Channel Discrimination



Channel Discrimination



Discrimination error with n channel uses

$$p_{\mathrm{err}} \leq \frac{1 - \frac{1}{2}\|\mathcal{E}_0^{\otimes n} - \mathcal{E}_1^{\otimes n}\|_\diamond}{2}$$

Easier bound

$$p_{\mathrm{err}} \leq \frac{F(\chi_{\mathcal{E}_0}, \chi_{\mathcal{E}_1})^n}{2} \qquad \chi_{\mathcal{E}_i} = \mathcal{I} \otimes \mathcal{E}_i[|\Phi\rangle\langle\Phi|]$$

Pirandola, Laurenza, Lupo, Pereira, npj:QI, 2019

Gaussian channel discrimination

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Optimal Measurement

$$E_x = |x\rangle\langle x| = \mathrm{eigenprojections}(\hat{M})$$

$$\hat{M} = \chi_{\mathcal{E}_0}^{-1/2} \sqrt{\chi_{\mathcal{E}_0}^{1/2} \chi_{\mathcal{E}_1} \chi_{\mathcal{E}_0}^{1/2}} \chi_{\mathcal{E}_0}^{-1/2}$$

Gaussian channel discrimination

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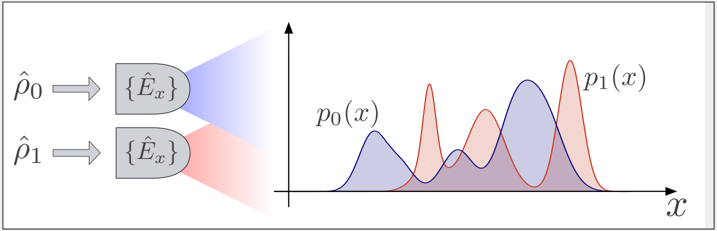
Continuous set of channels: “SLD measurement”

$$\hat{M}(\hat{\chi}_{\mathcal{E}_\theta}, \hat{\chi}_{\mathcal{E}_{\theta+d\theta}}) \simeq 1 - d\theta/2\hat{D}(u_\theta)(\hat{Q}^{\mathsf{T}} G_{\mathsf{M}} \hat{Q} - 2v_{\mathsf{M}}^{\mathsf{T}}\hat{Q})\hat{D}^{\dagger}(u_\theta)$$

$$4V_\theta G_{\mathsf{M}} V_\theta + \Omega G_{\mathsf{M}} \Omega + 2\frac{\partial V_\theta}{\partial \theta} = 0$$

Conclusions



- Optimal measurement to maximize the Bhattacharyya distance

$$\max_{E_x} D(p_0(x), p_1(x)) = \cos^{-1}\left[\mathrm{Tr}\sqrt{\sqrt{\rho_0}\rho_1\sqrt{\rho_0}}\right]$$

- Closed-form expression for two arbitrary Gaussian states with arbitrary number of modes