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Quantum Illumination Radars:

Relaxed Assumptions and Limitations

Reference: A. Karsa, G. Spedalieri, Q. Zhuang, & SP, *Quantum Illumination with a generic Gaussian source*,

Phys. Rev. Research 2, 023414 (2020)

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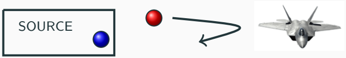
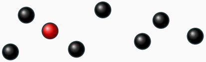
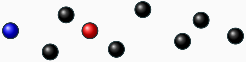
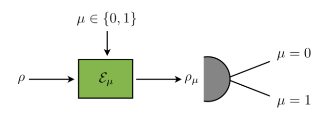
Literature

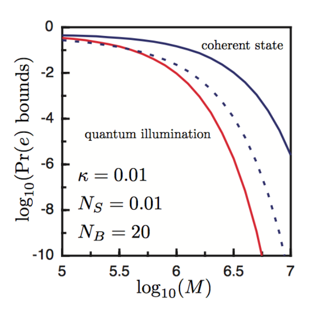
- Review on photonic quantum sensing covering the protocol of Gaussian quantum illumination (optical/microwave):
- **Advances in photonic quantum sensing**
  S. Pirandola, B.R. Bardhan, T. Gehring, C. Weedbrook, S. Lloyd
  Nature Photonics 12, 724-733 (2018)
- Has discussion on challenges and limits of quantum lidar/radar.

Basic QI protocol

1	Entangle Signal and Idler mode pairs	
2	Send Signal, retain Idler	
3	Collect Return = Noise + Signal (?)	
4	Measure Idler with Return	
5	Detect:	(i) Correlations ⇒ Target Present (ii) No correlations ⇒ Target Absent

Quantum hypothesis testing





- Quantum radar detection can be modelled as a problem of binary QHT: task reduced to distinguishing output states ρ_μ.
- Requires $\rho_\mu^{\otimes M}$ with $M >> 1$

$$P_{\mathrm{err}}^{QI} = \frac{1}{2} \exp\left(-\frac{M\kappa N_S}{N_B}\right) \quad (1)$$

$$P_{\mathrm{err}}^{CS} = \frac{1}{2} \exp\left(-\frac{M\kappa N_S}{4N_B}\right) \quad (2)$$

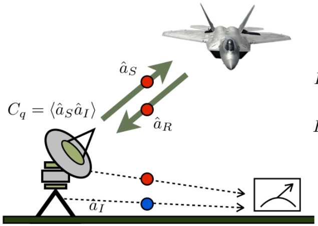
What we do...

- Generalise the definition of a quantum radar beyond QI:

  Any model that exploits a quantum part or device to outperform a corresponding classical radar under the same conditions of energy, range, etc.

- We progressively relax entanglement requirements of QI and study the corresponding detection performances to the point where the source becomes just-separable, i.e., a maximally-correlated separable state.
- At the same time, try to formulate equivalences between figures of merit between classical and quantum radar performance.

General quantum-correlated source



$$H_0 : \hat{a}_R = \hat{a}_B, \quad N_B >> 1$$

$$H_1 : \hat{a}_R = \sqrt{\kappa}\hat{a}_S + \sqrt{1-\kappa}\hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1-\kappa}$$

We consider a source modelled as a two-mode Gaussian state:

$$\mathbf{V}_{SI}^{gen} = \begin{pmatrix} S & C \\ C & S \end{pmatrix} \oplus \begin{pmatrix} S & -C \\ -C & S \end{pmatrix}, \tag{3}$$

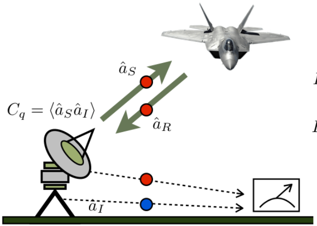
where

$$S := N_S + 1/2,$$

$$0 \leq C \leq \sqrt{S^2 - 1/4} = \sqrt{N_S(N_S+1)} = C_q,$$

$$C_d = N_S.$$

General quantum-correlated return



$$H_0 : \hat{a}_R = \hat{a}_B, \quad N_B >> 1$$

$$H_1 : \hat{a}_R = \sqrt{\kappa}\hat{a}_S + \sqrt{1-\kappa}\hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1-\kappa}$$

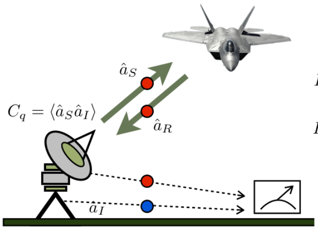
The joint state of our returning (R) mode and the retained idler is given by, under H_0 and H_1, respectively:

$$\mathbf{V}_{RI}^{(0)} = \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} \oplus \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix}, \tag{4}$$

$$\mathbf{V}_{RI}^{(1)} = \begin{pmatrix} A & \sqrt{\kappa}C \\ \sqrt{\kappa}C & S \end{pmatrix} \oplus \begin{pmatrix} A & -\sqrt{\kappa}C \\ -\sqrt{\kappa}C & S \end{pmatrix}, \tag{5}$$

where $B := N_B + 1/2$ and $A := \kappa N_S + B$.

Coherent state scenario - no access to idler



$$H_0 : \hat{a}_R = \hat{a}_B, \quad N_B >> 1$$

$$H_1 : \hat{a}_R = \sqrt{\kappa}\hat{a}_S + \sqrt{1-\kappa}\hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1-\kappa}$$

- Signal mode, annihilation operator $\hat{a}_S$, prepared in the coherent state $|\sqrt{N_S}\rangle$
- H_0: return is in a thermal state with mean photons per mode N_B, mean vector of zero and CM $(N_B + 1/2)\mathbf{1}_2$.
- H_1: return corresponds to a displaced thermal state with mean vector $(\sqrt{2\kappa N_S}, 0)$ and CM $(N_B + 1/2)\mathbf{1}_2$.

Hypothesis testing for quantum radar detection

- Detection probability $P_d := P(H_1|H_1)$.
- Two types of error may occur:
  type-I (false-alarm) $P_{fa} = P(H_1|H_0)$, and
  type-II (missed detection) $P_{md} = P(H_0|H_1)$.
- One may consider *asymmetric* testing in order to take in account discrepancies in error cost.
- A *symmetric* approach aims to obtain a global minimisation over all errors, irrespective of their origin. In this case, one considers the minimization of the average error probability

$$P_{\mathrm{err}} := P(H_0)P(H_1|H_0) + P(H_1)P(H_0|H_1). \tag{6}$$

Symmetric hypothesis testing

- *Minimum* error probability is given by the Helstrom bound: $P_{\text{err}}^{\min} = [1 - D(\hat{\rho}_0, \hat{\rho}_1)]/2$, where $D(\hat{\rho}_0, \hat{\rho}_1) := |\hat{\rho}_0 - \hat{\rho}_1|/2$ is the trace distance.
- Analytically, use QCB,

$$P_{\text{err}}^{\min} \leq P_{\text{err}}^{\text{QCB}} := \frac{1}{2}\left(\inf_{0\leq s\leq 1} C_s\right), \quad C_s := \left(\hat{\rho}_0^s \hat{\rho}_1^{1-s}\right). \tag{7}$$

- Forgoing minimization, set $s = 1/2$ and define a simpler, though weaker, upper bound, the QBB

$$P_{\text{err}}^{\text{QBB}} := \frac{1}{2}\left(\sqrt{\hat{\rho}_0}\sqrt{\hat{\rho}_1}\right). \tag{8}$$

- For Gaussian states, closed analytical formulae exist for these!

Symmetric radar detection

- Assume typical condition of QI: $\kappa << 1$, $N_B >> 1$, $N_S << 1$.
- For a TMSV state, the minimum error probability satisfies

$$P_{\text{err}}^{\text{TMSV}} \leq e^{-M\kappa N_S/N_B}/2. \quad (9)$$

- Computed using the QBB, exponentially tight in limit of large M.
- Error-rate exponent has a factor of 4 advantage over the corresponding coherent-state transmitter

$$P_{\text{err}}^{\text{CS}} \leq e^{-M\kappa N_S/4N_B}/2. \quad (10)$$

- Extending Eq. (9) to the error probability for a generic source, we find

$$P_{\text{err}}^{\text{gen}} \leq e^{-M\kappa N_S C^2/N_B C_q^2}/2. \quad (11)$$

Symmetric radar detection - generic source

- Comparing Eqs. (11) and (10), we see that a quantum-correlated transmitter beats the coherent state transmitter if $P_{\text{err}}^{\text{gen}} \leq P_{\text{err}}^{\text{CS}}$ which means

$$\frac{C^2}{C_q^2} \geq \frac{1}{4} \Rightarrow C \geq \frac{1}{2}\sqrt{N_S(N_S+1)}. \tag{12}$$

- Thus, the quadrature correlations required to outperform the semi-classical benchmark is **half** the value of those of a TMSV state.
- The employment of a source at the separable limit is not capable of beating coherent states under symmetric testing.

Asymmetric hypothesis testing 1

- Consider M copies $\hat{\rho}_i^{\otimes M}$ of the state $\hat{\rho}_i$ encoding bit $i \in \{0, 1\}$.
- Binary outcome - two types of error, i.e., the type-I (false alarm) error

$$P_{\text{fa}} := P(H_1|H_0) = \left(E_1 \hat{\rho}_0^{\otimes M}\right), \tag{13}$$

and the type-II (missed detection) error

$$P_{\text{md}} := P(H_0|H_1) = \left(E_0 \hat{\rho}_1^{\otimes M}\right). \tag{14}$$

- These probabilities are dependent on the number M of copies and, for $M >> 1$, they both tend to zero, i.e.,

$$P_{\text{fa}} \simeq e^{-\alpha_R M}, \; P_{\text{md}} \simeq e^{-\beta_R M}, \tag{15}$$

where we define the 'error-exponents' or 'rate limits' as

$$\alpha_R = -\lim_{M \to +\infty} \frac{1}{M} \ln P_{\text{fa}}, \quad \beta_R = -\lim_{M \to +\infty} \frac{1}{M} \ln P_{\text{md}}. \tag{16}$$

Asymmetric hypothesis testing 2

- Place a relatively loose constraint $P_{\text{fa}} < \epsilon$ on the type-I error, allowing us more freedom to minimize P_{md}.
- Quantum Stein's lemma: given this constraint, QRE is the optimal decay rate for the type-II error probability

$$D\left(\hat{\rho}_0||\hat{\rho}_1\right) = [\hat{\rho}_0(\ln\hat{\rho}_0 - \ln\hat{\rho}_1)] \quad (17)$$

- Tracking the type-II error exponent to second order (in M) depth, that is to order $\sqrt{M}$, allows one to define the QRE variance

$$V\left(\hat{\rho}_0||\hat{\rho}_1\right) = [\hat{\rho}_0(\ln\hat{\rho}_0 - \ln\hat{\rho}_1)^2] - [D\left(\hat{\rho}_0||\hat{\rho}_1\right)]^2. \quad (18)$$

- Optimal type-II error probability, for sample size M:

$$P_{\text{md}} = \exp\left\{-\left[MD\left(\hat{\rho}_0||\hat{\rho}_1\right) + \sqrt{MV\left(\hat{\rho}_0||\hat{\rho}_1\right)}\Phi^{-1}(\epsilon) + \mathcal{O}(\log M)\right]\right\}, \quad (19)$$

where $\epsilon \in (0, 1)$ bounds P_{fa} and $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} dx \exp\left(-x^2/2\right)$.

Remarks on Literature

1. Asymmetric hypothesis testing: Helstrom's and Holevo's books
2. Second-order expansion: Ke Li, Ann. Statist. 42, 171-189 (2014).
3. Computation of the relative entropy:
   Pirandola, Laurenza, Ottaviani, Banchi,
   Nature Communications 8, 15043 (2017)
4. Computation of the relative entropy variance:
   Wilde, Tomamichel, Lloyd, Berta, PRL 119, 120501 (2017)

- Calculations in items 3 and 4 are based on the Gaussian tools/formulas developed in:
  Banchi, Braunstein, Pirandola, PRL 115, 260501 (2015)

Asymmetric radar detection

- We evaluate the QRE and QRE-variance to first order in N_B by taking an asymptotic expansion for large N_B while keeping N_S fixed. We obtain

$$D_{\text{gen}} := D\left(\hat{\rho}_{RI}^{(0)} || \hat{\rho}_{RI}^{(1)}\right) = \frac{\kappa C^2}{N_B} \ln\left(1 + \frac{1}{N_S}\right) + \mathcal{O}\left(N_B^{-2}\right), \quad (20)$$

$$V_{\text{gen}} := V\left(\hat{\rho}_{RI}^{(0)} || \hat{\rho}_{RI}^{(1)}\right) = \frac{\kappa C^2 (2N_S + 1)}{N_B} \ln^2\left(1 + \frac{1}{N_S}\right) + \mathcal{O}\left(N_B^{-2}\right). \quad (21)$$

- For coherent states these quantities take the form

$$D_{\text{CS}} := D\left(\hat{\rho}_{\text{CS}}^{(0)} || \hat{\rho}_{\text{CS}}^{(1)}\right) = \kappa N_S \ln\left(1 + \frac{1}{N_B}\right) \simeq \gamma + \mathcal{O}\left(N_B^{-2}\right), \quad (22)$$

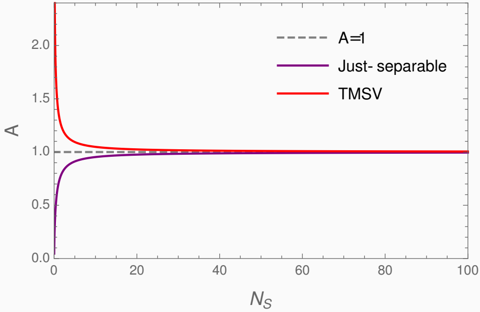
$$V_{\text{CS}} := V\left(\hat{\rho}_{\text{CS}}^{(0)} || \hat{\rho}_{\text{CS}}^{(1)}\right) = \kappa N_S (2N_B + 1) \ln^2\left(1 + \frac{1}{N_B}\right) \simeq 2\gamma + \mathcal{O}\left(N_B^{-2}\right), \quad (23)$$

- Signal-to-noise ratio (SNR): $\gamma := \frac{\kappa N_S}{N_B}$, usually expressed in decibels (dB) via $\gamma_{\text{dB}} = 10 \log_{10} \gamma$.

Asymmetric performance comparison

Define an error-exponent advantage over coherent states:

$$A(C, N_S) := \frac{D_{\text{gen}}}{D_{\text{CS}}} = \frac{C^2}{N_S} \ln\left(1 + \frac{1}{N_S}\right). \tag{24}$$



- Benefits of max-entanglement for QI only for very small energies.
- For increasing N_S, the ratio $A \to 1$, irrespective of source specification.
- Just-separable source quickly approaches the performance of QI already at about 20 photons.

Receiver operating characteristic

Study mis-detection probability vs false-alarm probability for generic Gaussian source and coherent state classical benchmark.

- Optimal: phase is maintained - homodyne + coherent integration
- Deterministic phase shift - use heterodyne + coherent integration.
- Random phase shift - heterodyne + non-coherent integration, given by Marcum's Q-function
- Marcum's Q -function may be overestimated by assuming single coherent pulse with MN_S photons.

ROC: generic quantum source

- The ROC $P_{\text{md}} = P_{\text{md}}(P_{\text{fa}})$ of the gen. quantum source can be upper bounded:

$$P_{\text{md}} \leq \tilde{P}_{\text{md}}^{\text{gen}} = \exp\left\{-\left[\sqrt{\frac{M\gamma}{N_S}}\Lambda C \ln\left(1+\frac{1}{N_S}\right) + \mathcal{O}(N_B^{-1}, 1)\right]\right\}, \tag{25}$$

$$\Lambda := \left(\sqrt{\frac{M\gamma}{N_S}} C + \sqrt{2N_S+1}\Phi^{-1}(P_{\text{fa}})\right). \tag{26}$$

- Sufficiently large M ($\gtrsim 10^7$) and large N_B ($\gtrsim 10^2$).

ROC: coherent states

- Optimal - homodyne + coherent integration and binary testing:

$$P^{\text{fa}}_{\mathrm{CS},\text{hom}}(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{M(2N_B+1)}}\right), \tag{27}$$

$$P^{\text{md}}_{\mathrm{CS},\text{hom}}(x) = \frac{1}{2}\,\text{erfc}\left(\frac{M\sqrt{2\kappa N_S}-x}{\sqrt{M(2N_B+1)}}\right), \tag{28}$$

where $\text{erfc}(z) := 1 - 2\pi^{-1/2}\int_0^z \exp(-t^2)dt$ is the complementary error function.

- Lower bound to non-coherent integration: assume single coherent state with mean number of photons MN_S so that the total SNR is given by $M\gamma$:

$$P^{\text{Marcum}}_{\text{md}} = 1 - Q\left(\sqrt{2M\gamma}, \sqrt{-2\ln P_{\text{fa}}}\right), \tag{29}$$

where the Marcum Q-function is defined as

$$Q(x,y) := \int_y^\infty dt\; te^{-(t^2+x^2)/2} I_0(tx). \tag{30}$$

Parameters

Focusing on short-range ($\sim$ 1m) applications, e.g. security or biomedical:

- For $\nu = 1$GHz (L band) and $T = 290\mathrm{K}$ (room temperature), we get $N_B \simeq 6 \times 10^3$ photons (bright noise)
- Assume broadband pulses, with 10% bandwidth (100MHz), so that their individual duration $\sim$ 10ns. Using $M = 10^8$ pulses then we have an integration time $\sim$ 1s - acceptable for slow-moving/still objects.
- Low-energy applications - assume $N_S = 1$ mean photon per pulse.
- What about the SNR γ? - Related to overall reflectivity κ, estimated by the radar equation.

Parameters

- The radar equation relates returning signal power P_R to the transmitted signal power P_T:

$$P_R = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4} P_T \tag{31}$$

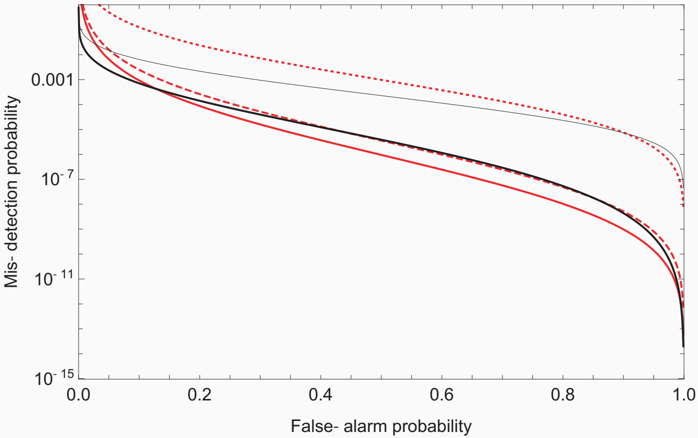
- At the same time, we can also write κ as the ratio:

$$\kappa = \frac{P_R}{P_T} = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4}, \tag{32}$$

- Assume $F = 1$ (no free-space loss) and ideal pencil beam so that solid angle δ is exactly subtended by the target's σ (valid at short range) so that $G = 4\pi/\delta = 4\pi R^2/\sigma$.
- Then,

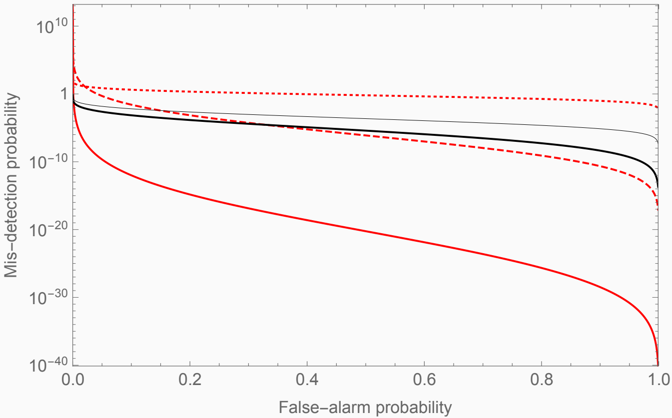
$$\kappa = \frac{A_R}{(4\pi R)^2}, \quad R = \frac{1}{4\pi}\sqrt{\frac{A_R}{\kappa}}. \tag{33}$$

ROC comparison: $N_S = 1, R = 1m$



- Red curves: Gaussian state QI with $C(p) = pC_d + (1-p)C_q$.
  Just-separable, $p = 0$, (dotted), maximal entanglement ($p = 1$), solid, and intermediate correlations ($p = 1/6$), dashed.
- Black curves: Classical coherent state benchmark
  Optimal homodyne detection, thick, and Marcum bound, thin.

ROC comparison: $N_S = 0.01, R = 0.1m$



- Red curves: Gaussian state QI with $C(p) = pC_d + (1-p)C_q$. Just-separable, $p = 0$, (dotted), maximal entanglement ($p = 1$), solid, and intermediate correlations ($p = 1/2$), dashed.
- Black curves: Classical coherent state benchmark Optimal homodyne detection, thick, and Marcum bound, thin.

Concluding remarks

- We have investigated how to loosen QI transmitter requirements.
- Scenarios of symmetric and asymmetric testing where we test the quantum performance with respect to suitable classical benchmarks.
- Quantum advantage still exists by using Gaussian sources which are not necessarily maximally entangled.
- Short ranges only (so spherical beam spreading does not involve too many dBs of loss, a major killing factor for any quantum radar design based on the exploitation of quantum correlations)
- A short-range, low-power radar is potentially interesting as noninvasive scanning tool for biomedical applications but also for short-range security and safety purposes.