Quantum state and channel discrimination

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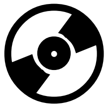
Brief Outline

- **Quantum State Discrimination**
- Helstrom Bound
- Fidelity, Quantum Chernoff Bound
- **Quantum Channel Discrimination**
- Formulation and Open Problems
- Discrimination of Bosonic Lossy Channels
- Application to Readout of Memories: **Quantum Reading** 

Quantum State Discrimination

Suppose we are given a quantum system which can be in two possible quantum states with the same probability

Binary Test $\rho \begin{cases} \rho_0 \\ \rho_1 \end{cases}$

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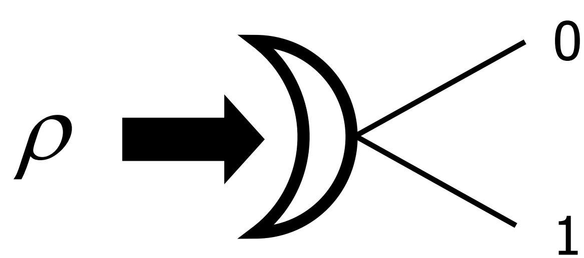

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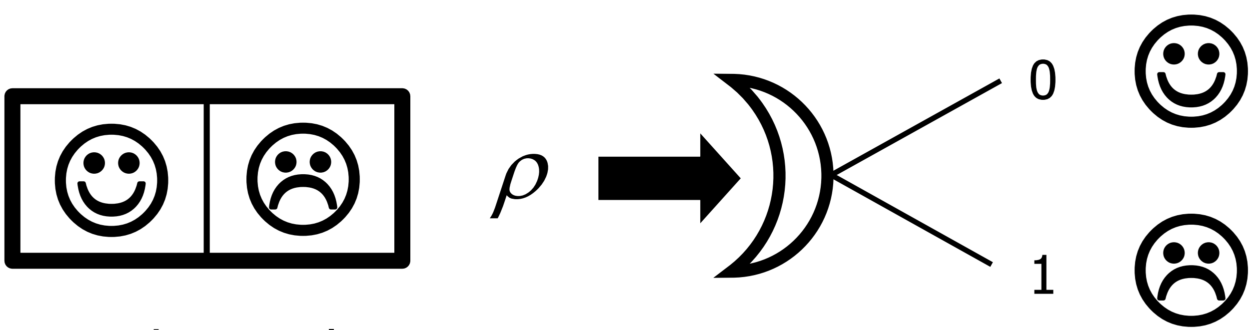

Orthogonal

Perfect discrimination

Quantum State Discrimination

Suppose we are given a quantum system which can be in two possible quantum states with the same probability

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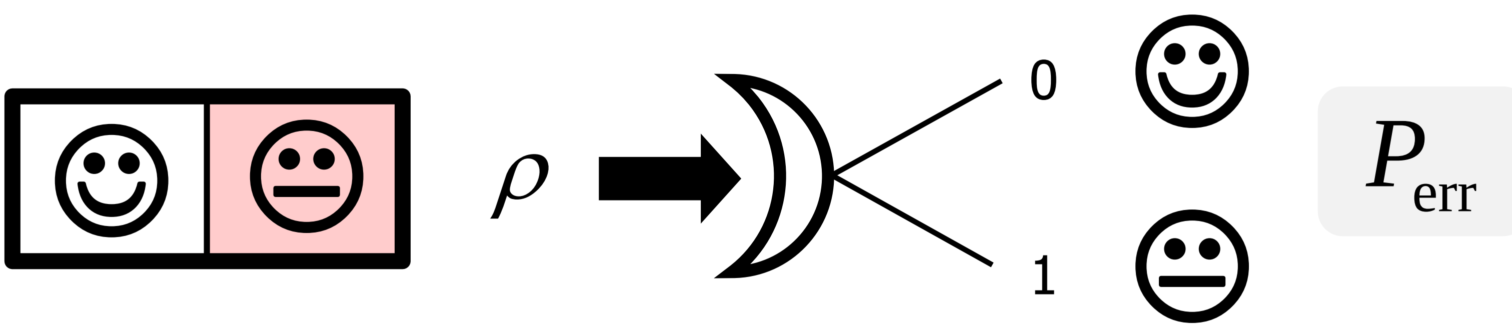

Non-orthogonal

Imperfect discrimination

Quantum State Discrimination

Suppose we are given a quantum system which can be in two possible quantum states with the same probability

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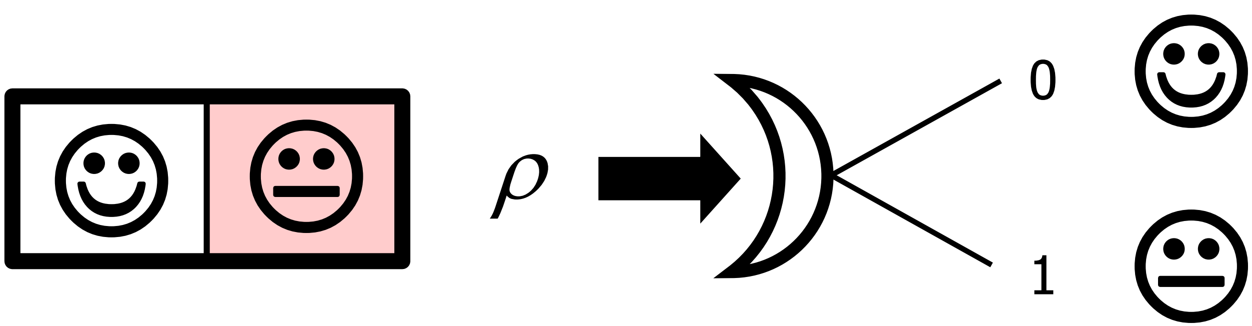

P_{err}

What is the **minimum error probability** in the statistical discrimination**?**

Quantum State Discrimination: General Solution

- Minimum Error Probability (Helstrom Bound)

$$P_{\text{err}} = \tfrac{1}{2}[1 - \underbrace{D(\rho_0, \rho_1)}_{\tfrac{1}{2}\mathrm{Tr}\,|\rho_0 - \rho_1|}]$$

(Trace distance)

Quantum State Discrimination: General Solution

- Minimum Error Probability (Helstrom Bound)

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(Trace distance)

- Trace distance is not easy to compute (difference between density operators)

Quantum State Discrimination: General Solution

- Minimum Error Probability (Helstrom Bound) $P_{\text{err}} = \frac{1}{2}[1 - D(\rho_0, \rho_1)]$

- Trace distance is not easy to compute (<u>difference</u> between density operators)

$$D(\rho_0, \rho_1) = \frac{1}{2}\mathrm{Tr}\,|\rho_0 - \rho_1|$$

(Trace distance)

- Upper and lower bounds based on **powers** of density operators

⇨ Fidelity $F(\rho_0, \rho_1) = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\rho_0}\,\rho_1\sqrt{\rho_0}}\right)\right]^2$

⇨ Quantum Chernoff Bound $Q(\rho_0, \rho_1) = \inf_{s\in(0,1)} \mathrm{Tr}\left(\rho_0^s \rho_1^{1-s}\right)$

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$$\frac{1}{2}\mathrm{Tr}\,|\,\rho_0 - \rho_1\,|$$

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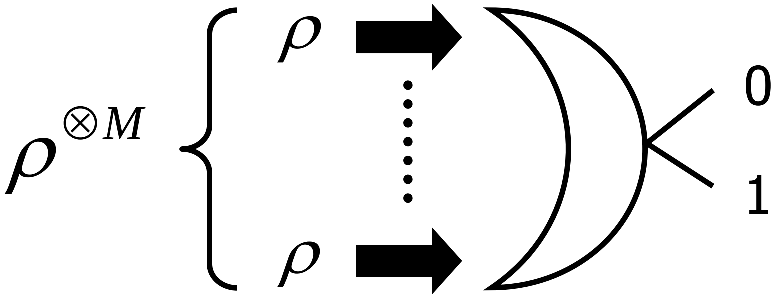
$$\frac{1-\sqrt{1-F}}{2} \le P_{\mathrm{err}} \le \frac{Q}{2}$$

[Fuchs and de Graaf, IEEE Trans. Inf. Theory 45, 1216 (1999)]
[Audenaert *et al.*, PRL 98, 160501 (2007)] [Pirandola and Lloyd, PRA78, 012331 (2008)]

M-copy Discrimination



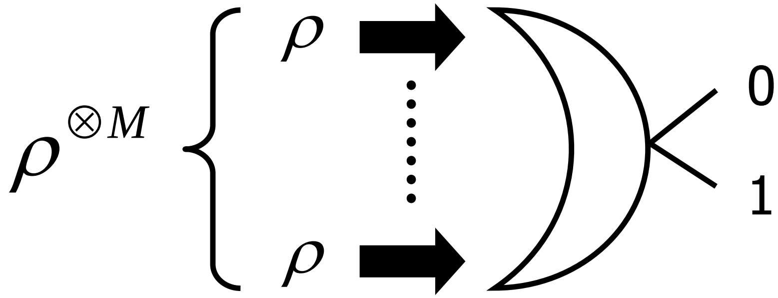
Fidelity and QCB are **multiplicative** under tensor products of density operators

$$F(\otimes M) = F^{M} \qquad Q(\otimes M) = Q^{M}$$

We can estimate the M-copy error probability using single-copy expressions

$$\frac{1-\sqrt{1-F^{M}}}{2} \leq P_{\mathrm{err}}^{(M)} \leq \frac{Q^{M}}{2}$$

M-copy Discrimination



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We can estimate the M-copy error probability using single-copy expressions

$$P_{\text{err}}^{(M)} \approx \frac{Q^M}{2} \qquad M \to \infty$$

QCB = good estimate of the min error prob.

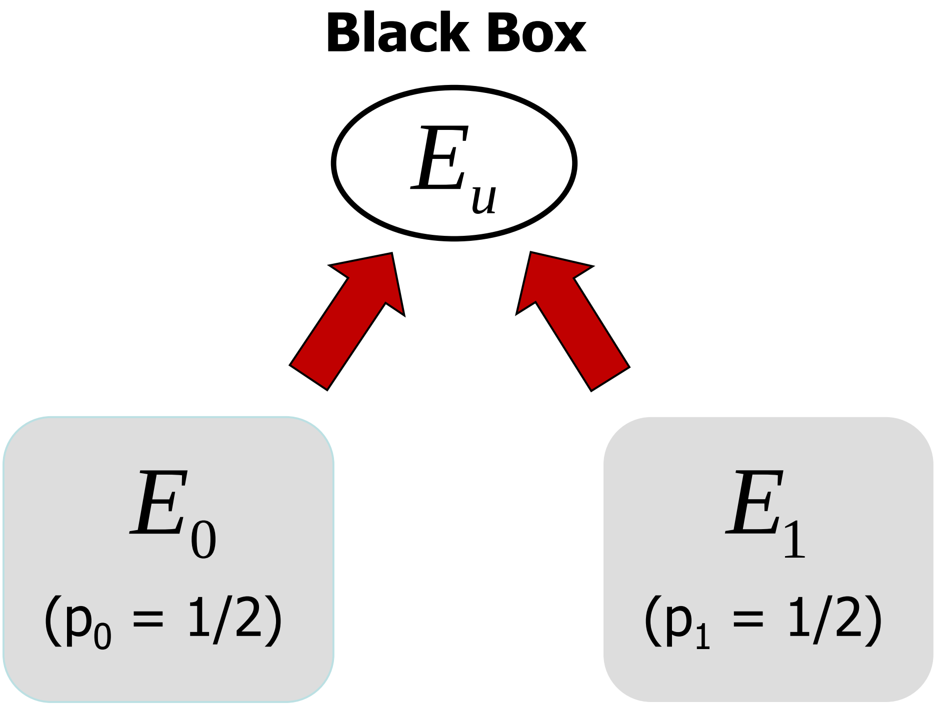
[Nussbaum and Szkola, Ann. Stat. 37, 1040 (2009)]
[Audenaert *et al.*, PRL 98, 160501 (2007)]

Quantum Channel Discrimination

And Many More!

Quantum Channel Discrimination

Black Box containing one of two channels with the same probability



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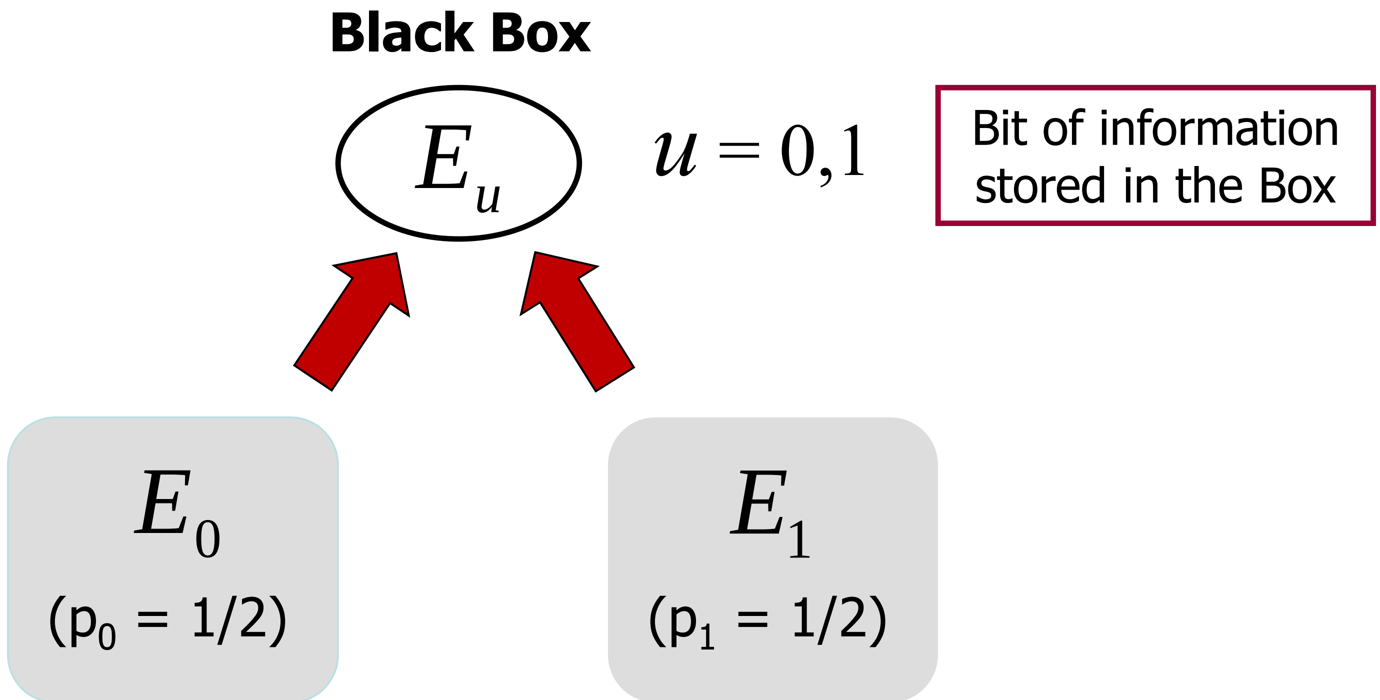

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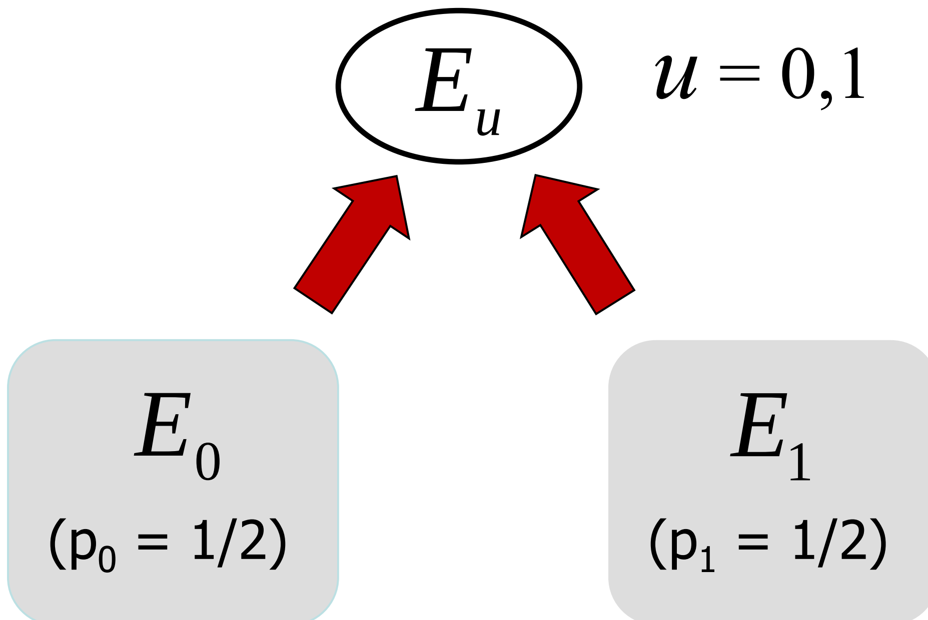
Black Box



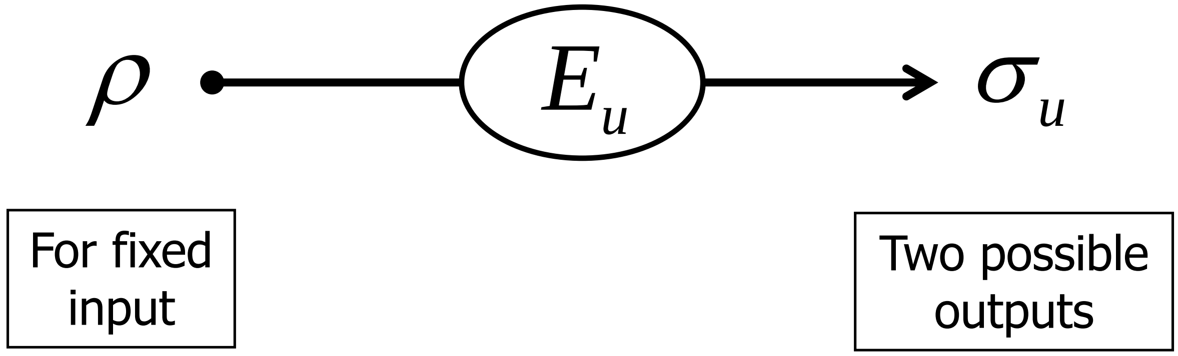
Bit of information stored in the Box

$$P_{\text{err}}(E_0 \neq E_1)\,?$$

Max Info retrievable from the Box?

Quantum Channel Discrimination

Retrieving the bit from the Box



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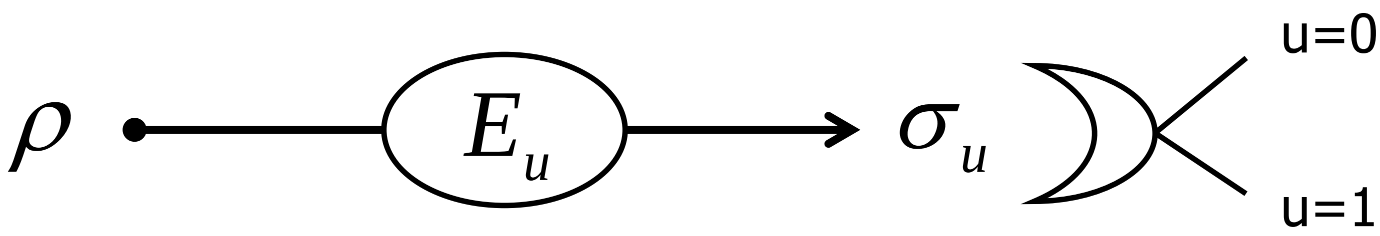

For fixed input

Two possible outputs

Optimal measurement (Helstrom)

$$P_{\text{err}}(\sigma_0 \neq \sigma_1) = \tfrac{1}{2}[1 - D(\sigma_0, \sigma_1)]$$

Quantum Channel Discrimination

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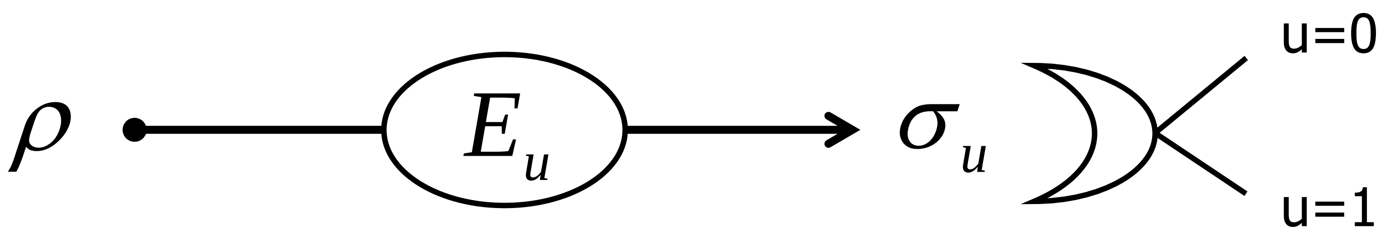

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$$\min_{\rho}\left[P_{\mathrm{err}}(\sigma_0 \neq \sigma_1)\right] = P_{\mathrm{err}}(E_0 \neq E_1)$$

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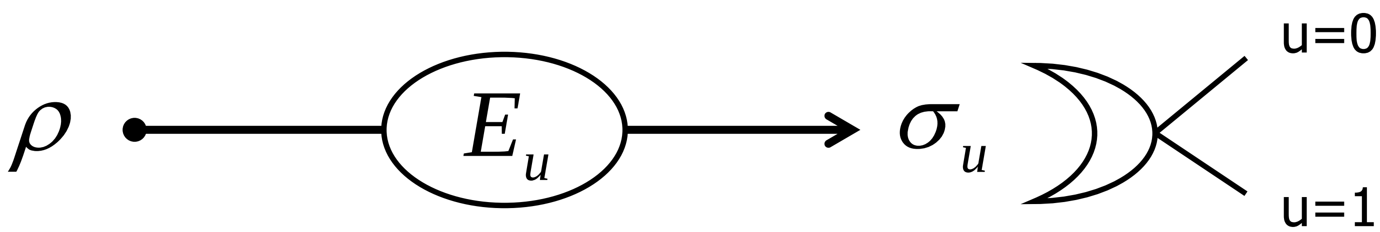


Open Problem

What is the optimal input?

Two possible outputs

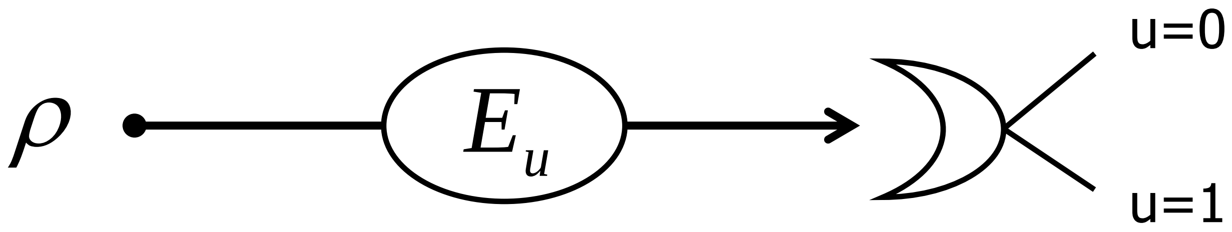
Optimal measurement (Helstrom)

$$P_{\text{err}}(\sigma_0 \neq \sigma_1) = \tfrac{1}{2}[1 - D(\sigma_0, \sigma_1)]$$

$$\min_{\rho}\left[P_{\text{err}}(\sigma_0 \neq \sigma_1)\right] = P_{\text{err}}(E_0 \neq E_1)$$

Quantum Channel Discrimination

General formulation of the problem



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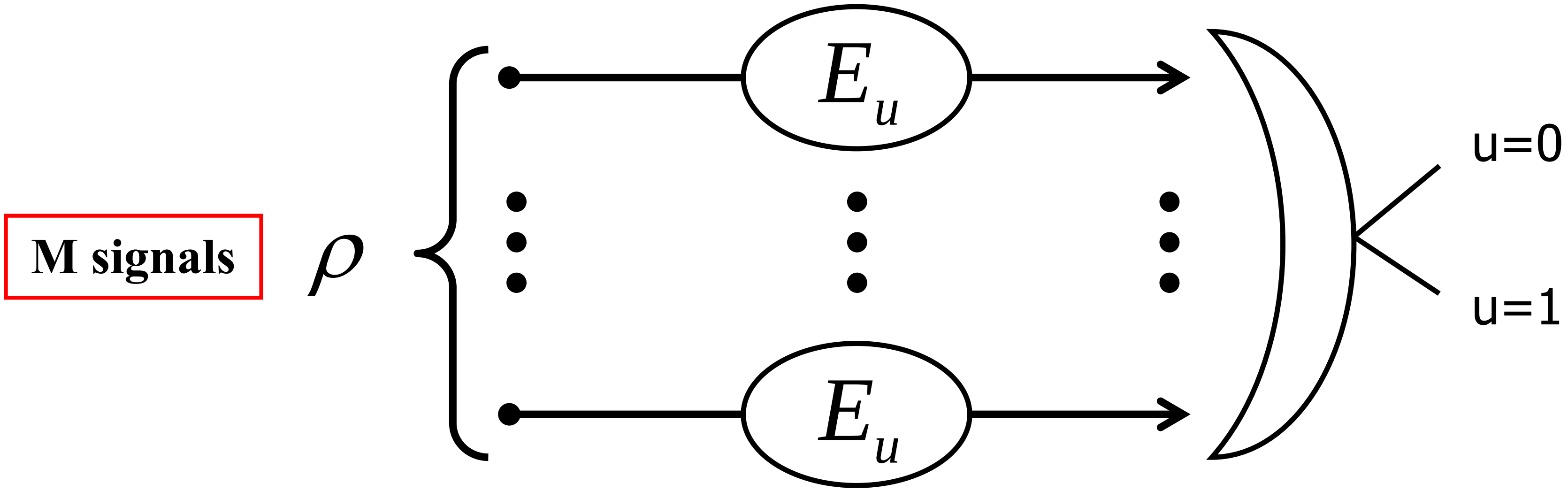

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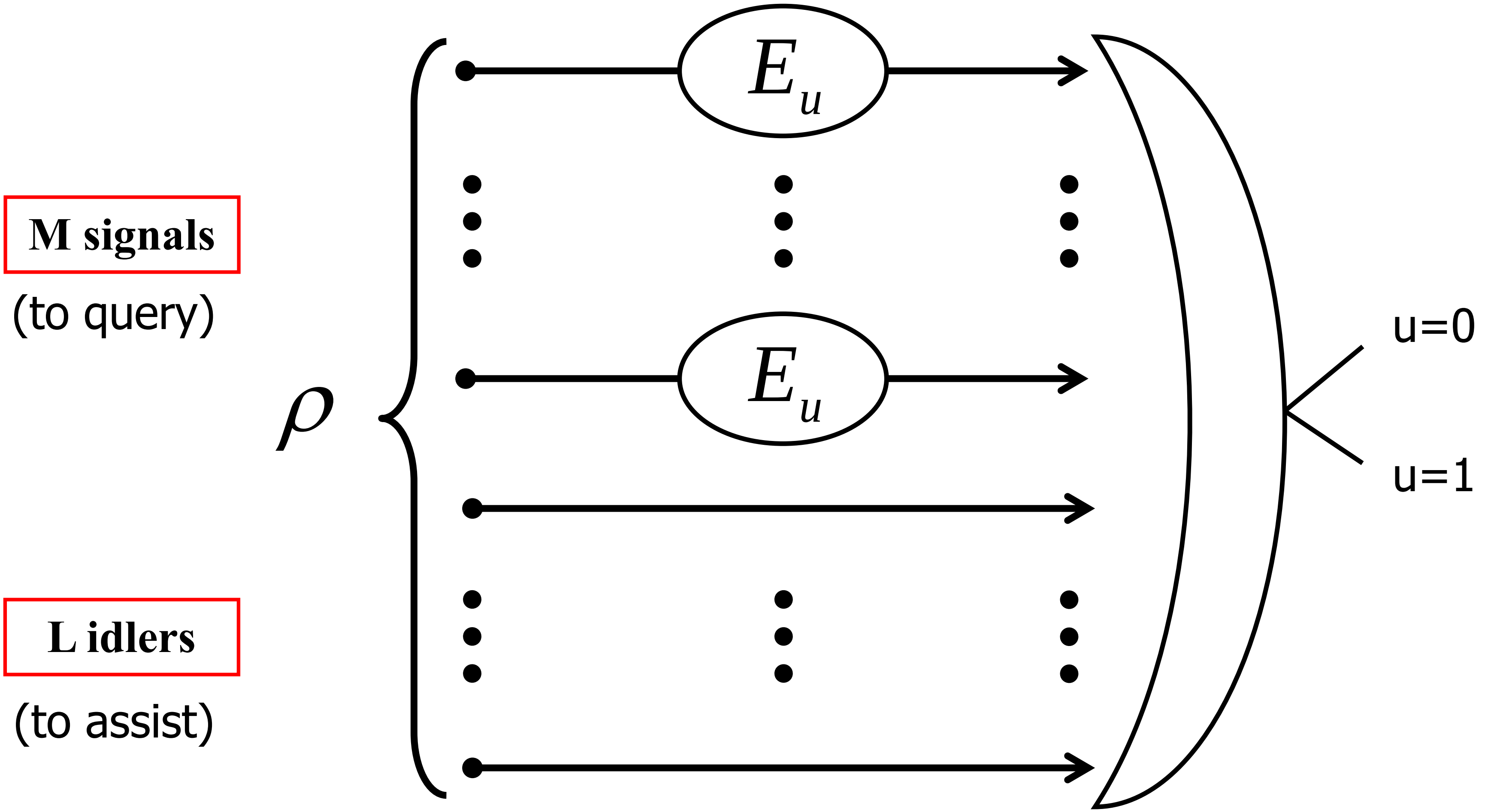
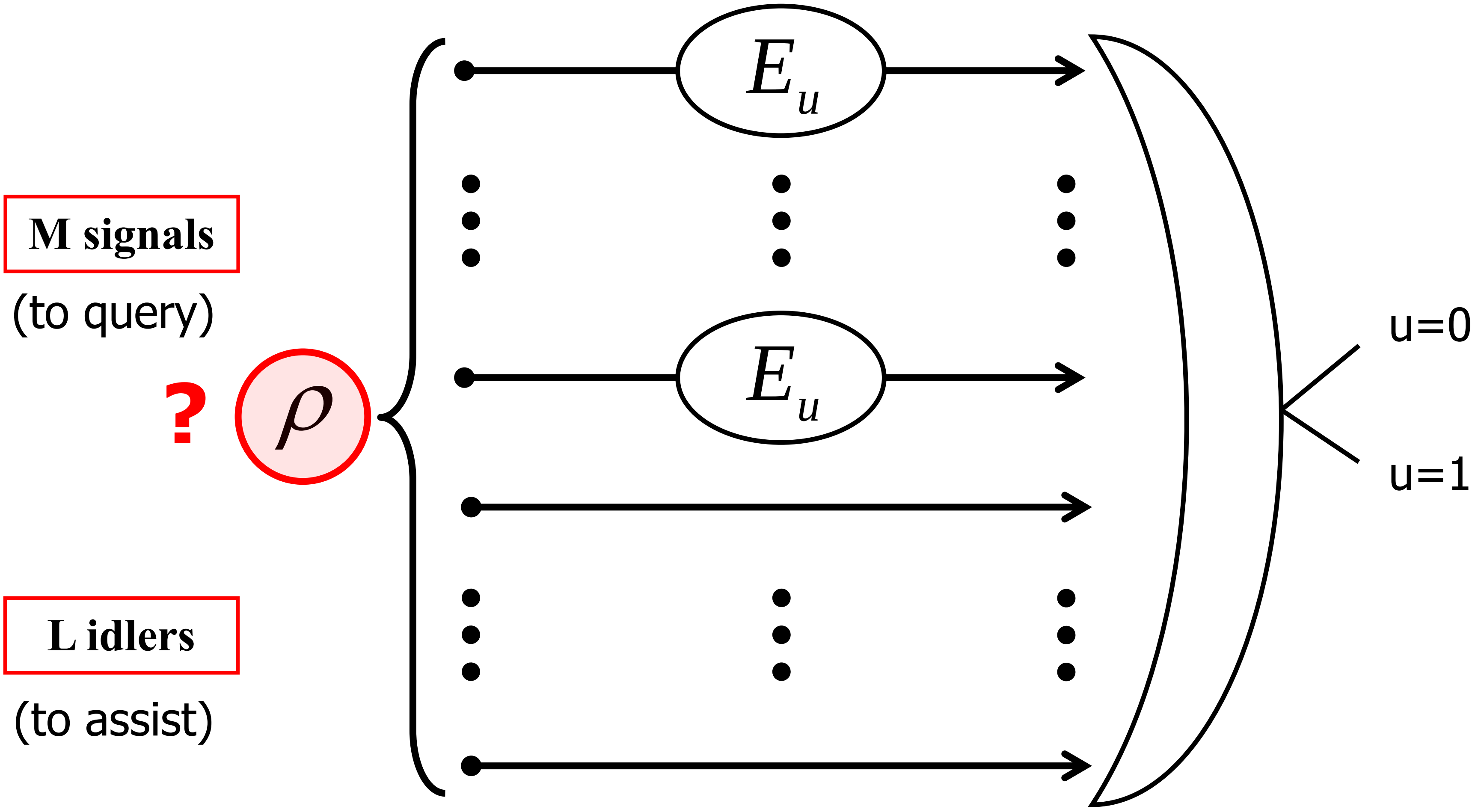


Quantum Channel Discrimination

Open problem: optimal input?



Discrimination of Bosonic Lossy Channels

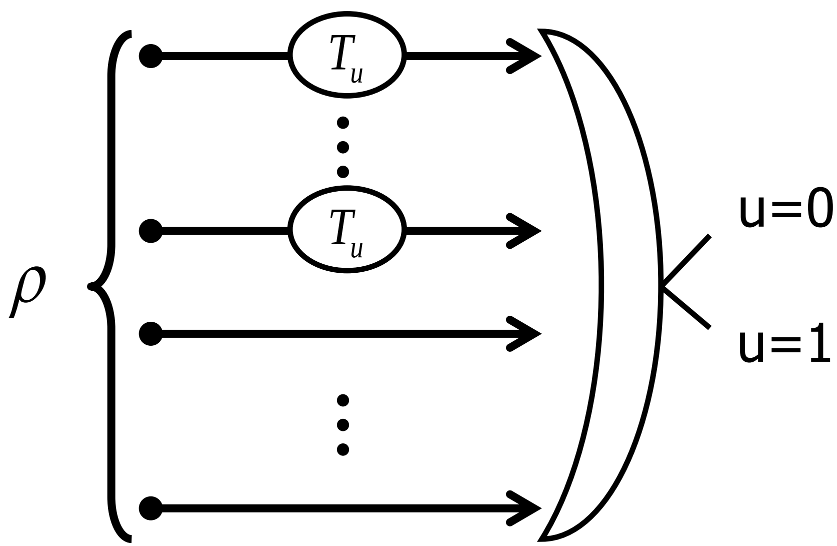
Simplest case: discrimination of lossy channels

www.opticalcables.org

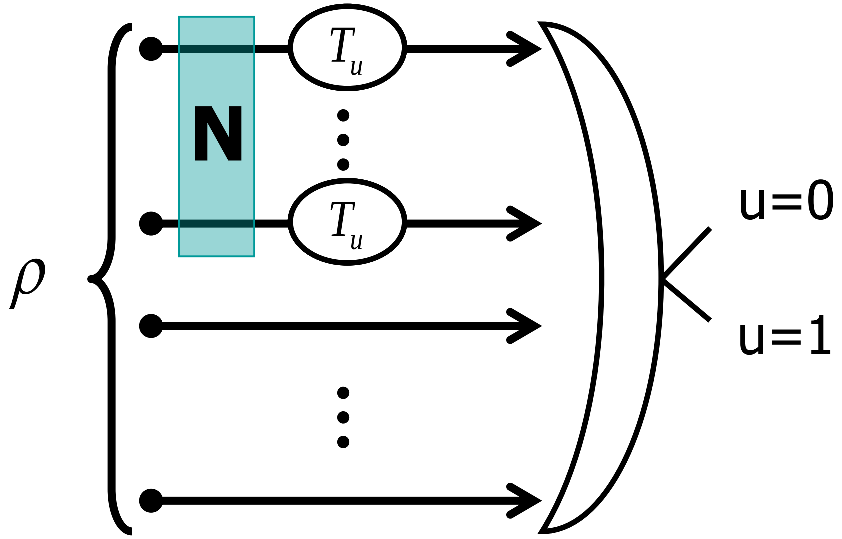
Lossy Channel $E = E(T)$

Transmissivity

$$E_0 \neq E_1 \Leftrightarrow T_0 \neq T_1$$



Discrimination of Bosonic Lossy Channels

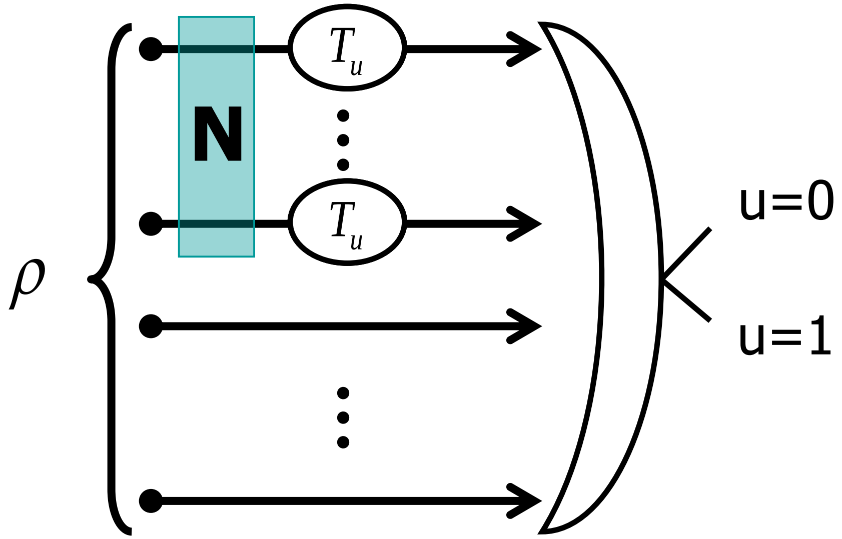


N = mean total number of photons irradiated over the box

We have $P_{\text{err}} \approx Exp\left(-|T_0 - T_1| N\right)$

Easy to find high-energy states ρ such that $P_{\text{err}} \rightarrow 0$

Discrimination of Bosonic Lossy Channels

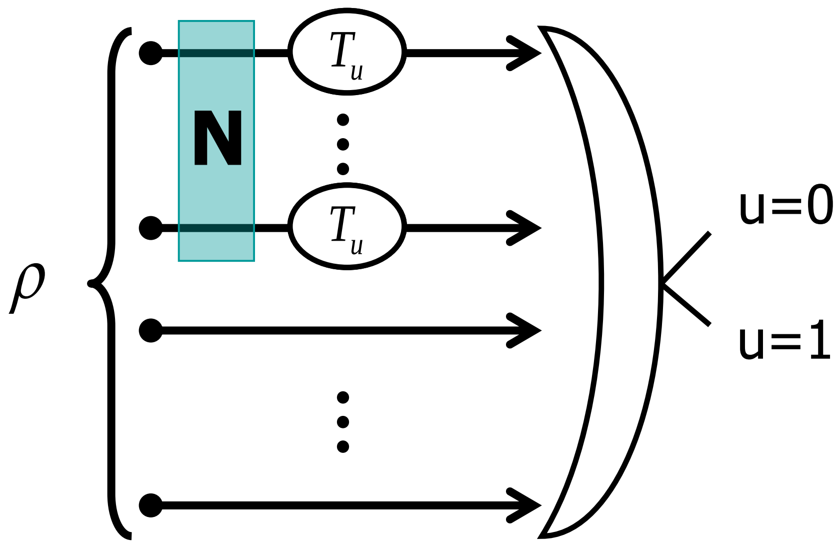


Constrained Optimization:

For fixed N, what is the optimal input state?

We can compare classes of states:
classical versus nonclassical

Discrimination of Bosonic Lossy Channels



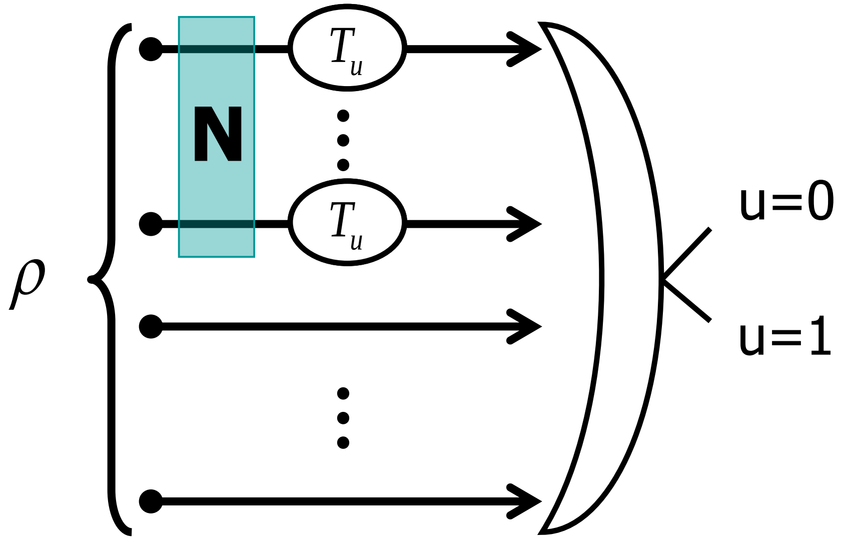
$$\rho = \int d^{2n}\alpha \, \wp(\alpha) |\alpha\rangle\langle\alpha|$$

(P-representation)

$\wp(\alpha) \geq 0$ **Classical state**

$\wp(\alpha) \not\geq 0$ **Nonclassical state**

Discrimination of Bosonic Lossy Channels



(Coherent states and their mixtures)

Classical state

$\wp(\alpha) \geq 0$

$$\rho = \int d^{2n}\alpha \, \wp(\alpha) |\alpha\rangle\langle\alpha|$$

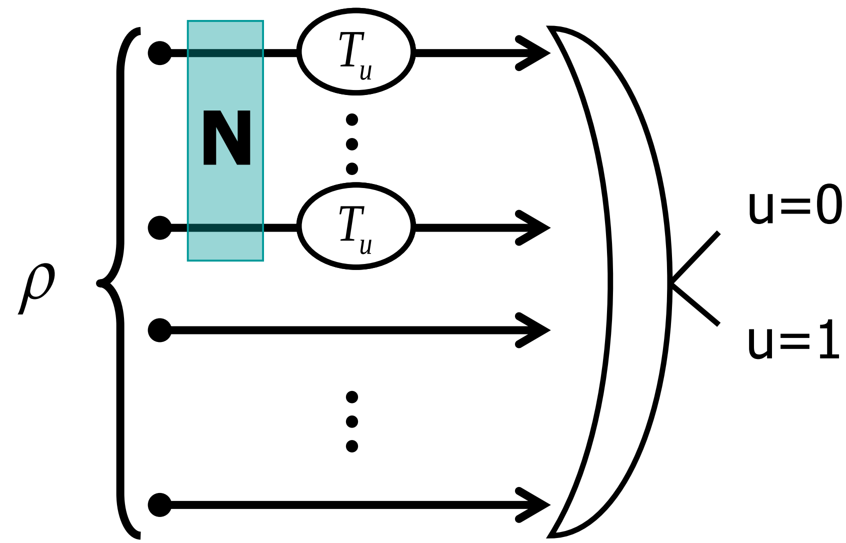
(P-representation)

$\wp(\alpha) \not\geq 0$

Nonclassical state

(Fock states, Squeezed states, Entangled states, NOON states)

Discrimination of Bosonic Lossy Channels

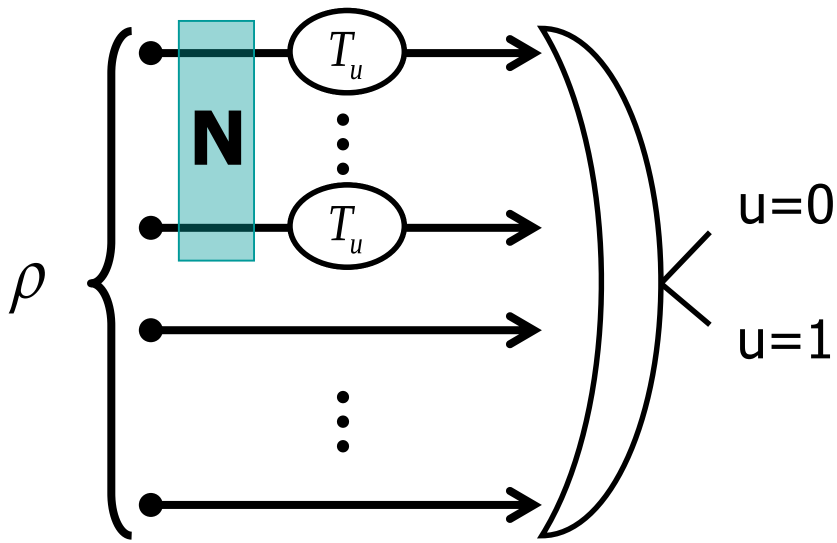


For fixed energy N:

Is there a nonclassical state which outperforms any classical state?

[S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011)]

Discrimination of Bosonic Lossy Channels



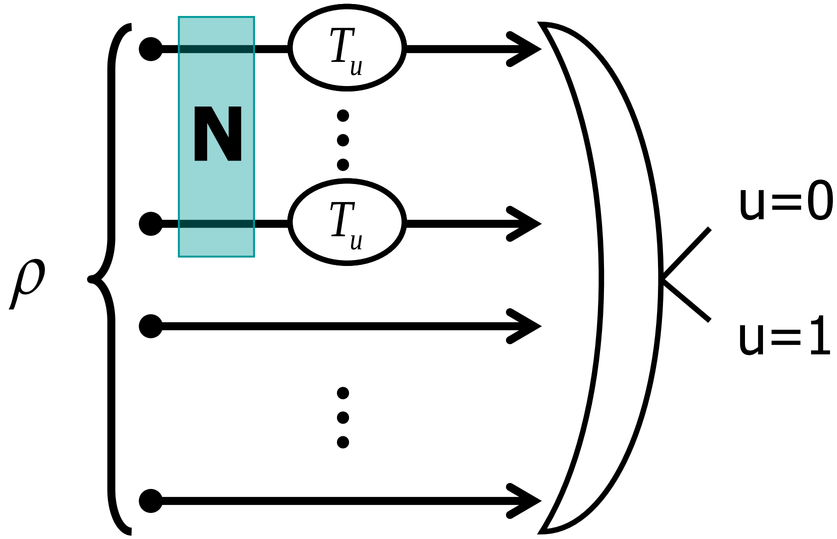
For fixed energy N:

Is there a nonclassical state which outperforms any classical state?

Yes! For $N > N_{th}(T_0, T_1) = \dfrac{2\log 2}{2 - T_0 - T_1 - 2\sqrt{(1-T_0)(1-T_1)}}$

[S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011)]

Discrimination of Bosonic Lossy Channels



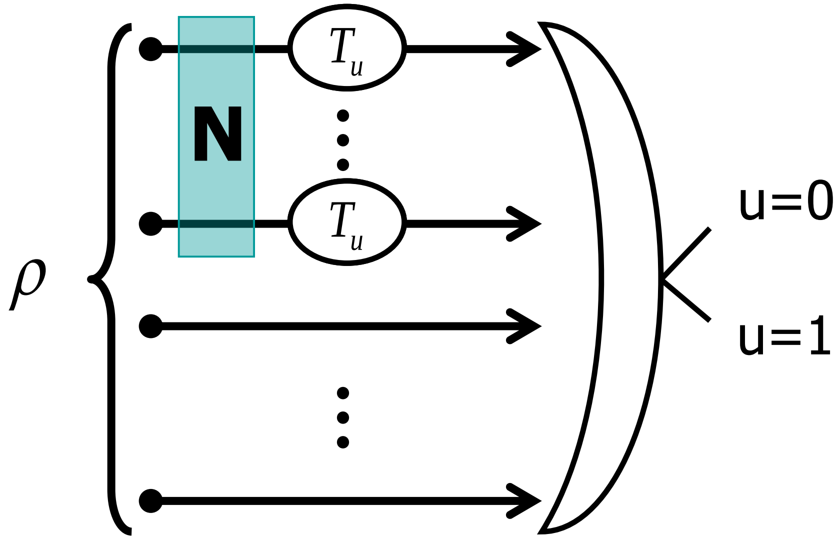
For fixed energy N:

Is there a nonclassical state which outperforms any classical state?

$N > N_{th} = 1/2$ for $\begin{cases} T_0 = T & \textit{Lossy channel} \\ T_1 = 1 & \textit{Ideal channel} \end{cases}$

For more than half photon, there is a nonclassical state which outperforms any classical state for detecting the presence of loss

Discrimination of Bosonic Lossy Channels



For fixed energy N:

How big is the advantage by using a nonclassical state?

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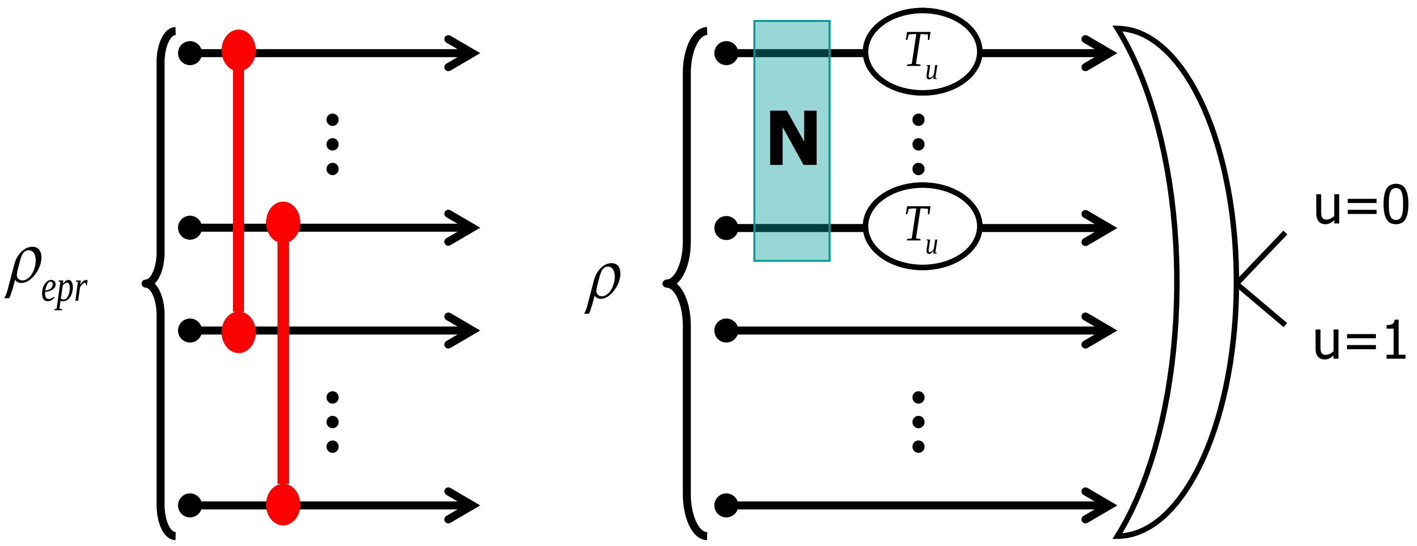
Discrimination of Bosonic Lossy Channels



For fixed energy N:

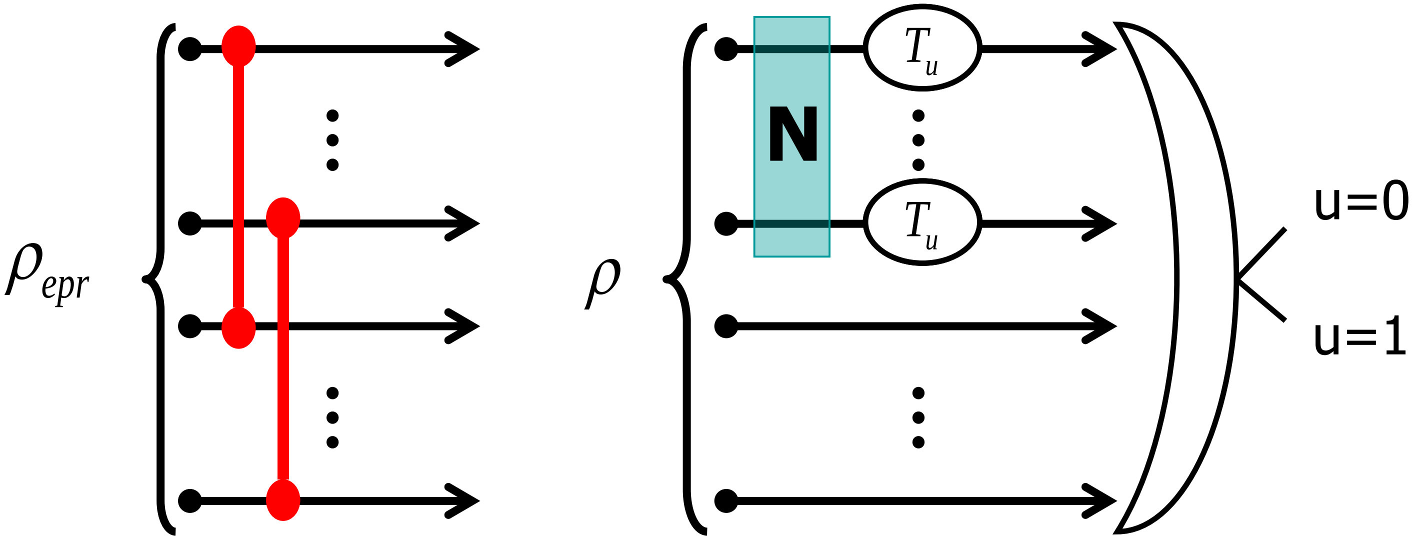
How big is the advantage by using a nonclassical state?

As non-classical state, let us consider an EPR source composed by tensor-product of two-mode squeezed vacuum states

$$\rho_{epr} = \rho_{TMSV} \otimes \cdots \otimes \rho_{TMSV}$$

[S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011)]

Discrimination of Bosonic Lossy Channels



For fixed energy N:

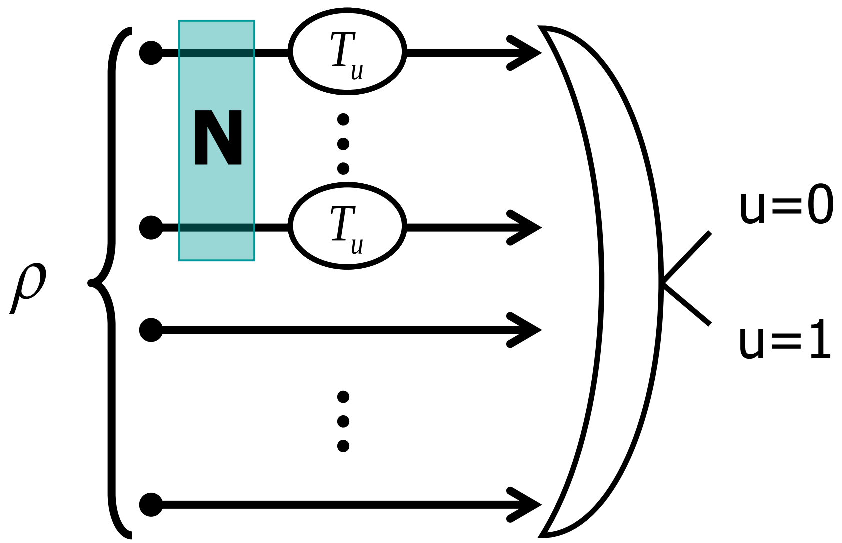
How big is the advantage by using a nonclassical state?

Information Gain

$$G = I(\rho_{epr}) - \sup_{\rho_{class}} I(\rho_{class})$$

[S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011)]

Discrimination of Bosonic Lossy Channels

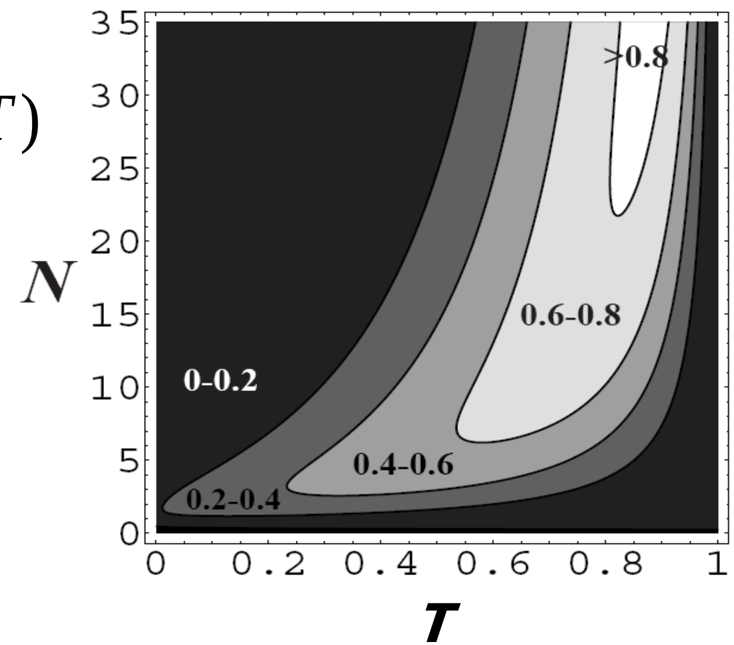


For detecting loss $\begin{cases} T_0 = T \\ T_1 = 1 \end{cases}$ we have: $G = G(N, T)$

We can have $G > 0$ for low energies

We can reach $G \approx 1$

In this case:
only nonclassical
states can work



Discrimination of Bosonic Lossy Channels

Take-home messages

Nonclassical states are able to outperform classical states in the discrimination of losses at fixed energy

The advantage can be remarkable in the regime of few photons

→ Identify states for the optimization problem

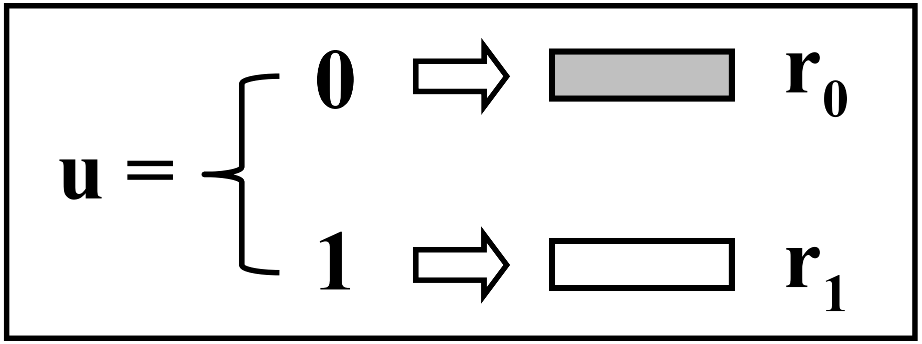
→ Application of nonclassical light in information technology

Quantum Reading of a Classical Digital Memory

[S. Pirandola, Phys. Rev. Lett. 106, 090504 (2011)]

Quantum Reading of a Classical Digital Memory

Logical bit stored in a reflecting medium with two possible reflectivities



Black-Box = Memory Cell

Quantum Reading of a Classical Digital Memory

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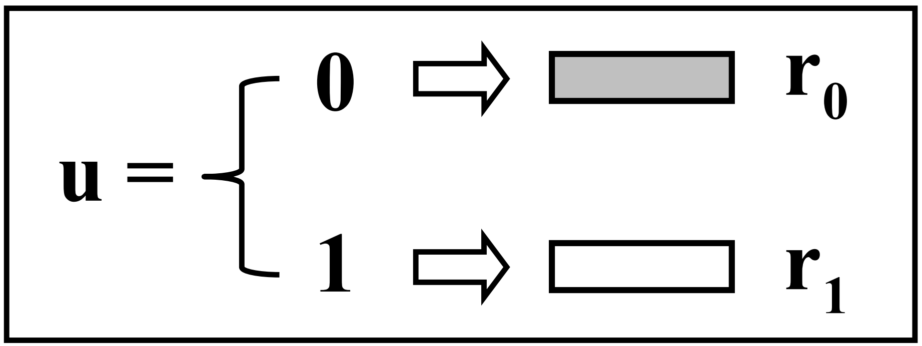
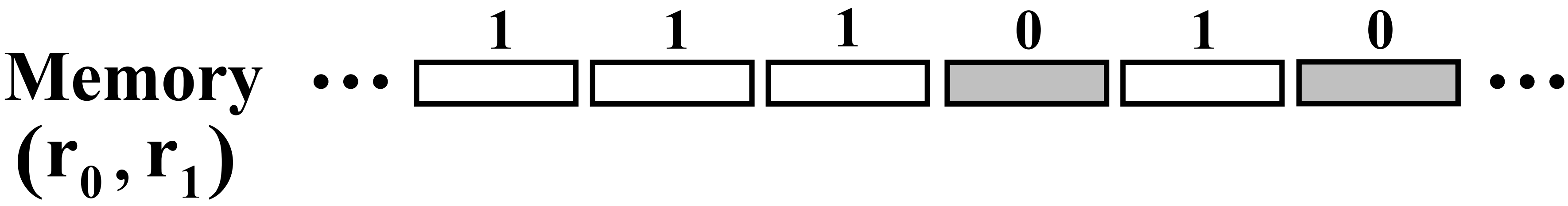


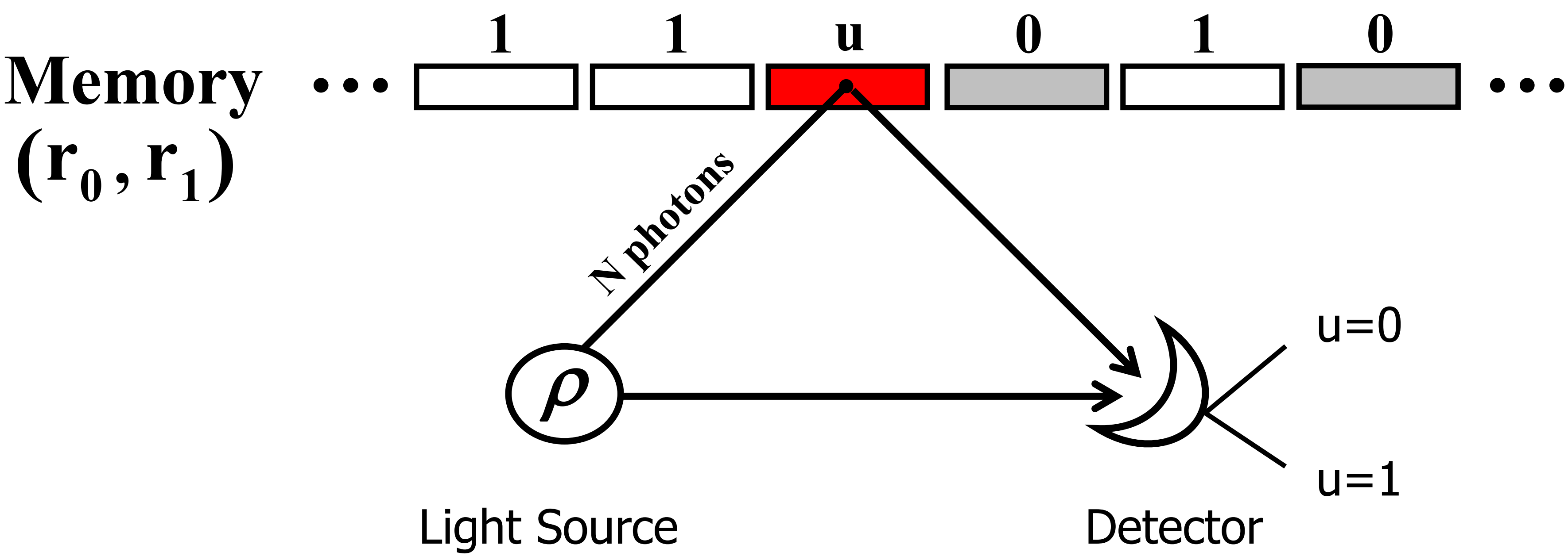


Digital memory whose cells are surfaces with different reflectivities

Quantum Reading of a Classical Digital Memory

General readout of a cell

$P_{err} \Rightarrow I$ (# bits decoded per cell)



Quantum Reading of a Classical Digital Memory

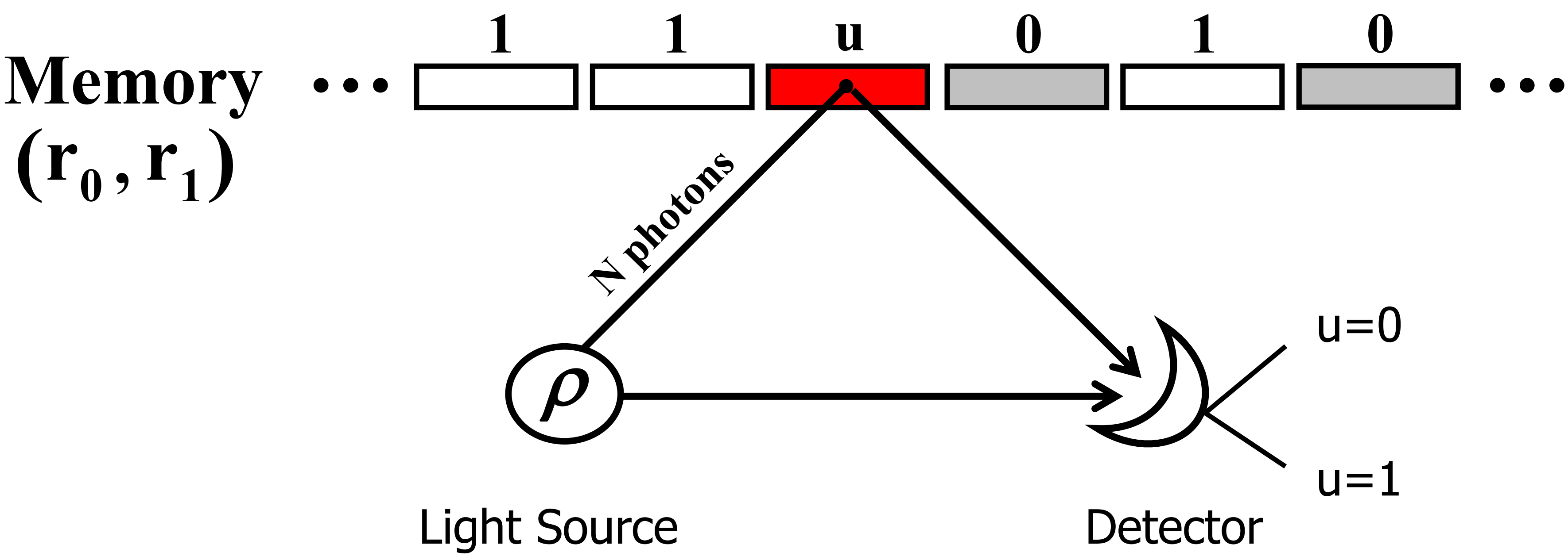
General readout of a cell

$$P_{err} \Rightarrow I$$

(# bits decoded per cell)

$$G = I(\rho_{epr}) - \sup_{\rho_{class}} I(\rho_{class})$$

(Bits per cell gained by quantum reading)



Quantum Reading of a Classical Digital Memory

General readout of a cell

$$P_{err} \Rightarrow I$$

(# bits decoded per cell)

$$G = I(\rho_{epr}) - \sup_{\rho_{class}} I(\rho_{class})$$

(Bits per cell gained by quantum reading)

For **few-photon signals** we can have:

$$G \to 1$$

Quantum reading:
1 bit per cell

Classical reading:
0 bit per cell

Extreme enhancement in the regime of few photons!

Quantum Reading of a Classical Digital Memory

Take-home message

Quantum reading is the best for **few photons**

→

What is the regime of **few photons** ?

Quantum Reading of a Classical Digital Memory

Take-home message

Quantum reading is the best for **few photons** → What is the regime of **few photons** ?

$$P = (h\nu)\frac{N}{\tau}$$

Laser Power → P

Light Frequency → ν

Photons → N

Pulse Duration → τ

- fewer photons = shorter pulses (e.g. 1 fs) → Faster data-transfer rates
- fewer photons = higher frequencies (e.g. UV light) → Higher storage capacities

Quantum Reading of a Classical Digital Memory

Improving the model

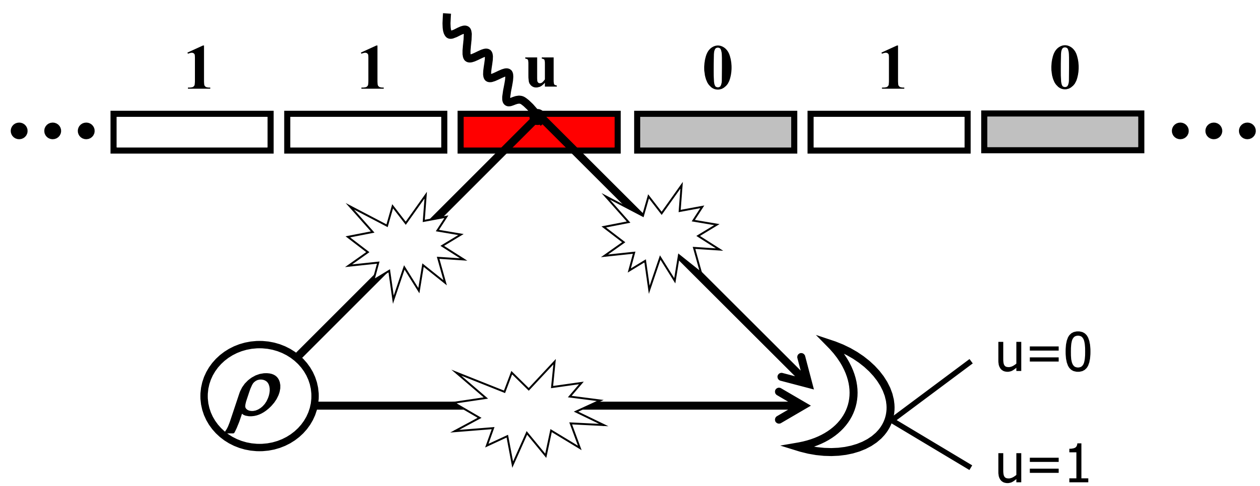
1) Internal thermal decoherence in the quantum reader **Checked**

2) Stray photons hitting the memory **Checked**

3) Error correcting codes with small overhead **Checked**

4) Practical sub-optimal receiver for detection **Checked**

5) Inter-bit interference caused by diffraction **Checked**



Thanks for your attention!!