Quantum illumination and quantum radar

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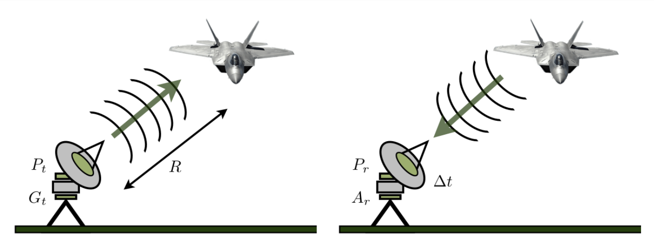
Introduction

- Quantum technologies has become a vast and highly active field of research: computation, cryptography, sensing, metrology...
- Making use of quantum features (superposition and entanglement) achieve otherwise impossible results, e.g., quadratic improvement in measurement sensitivities ($\propto 1/\sqrt{N} \rightarrow \propto 1/N$)
- Quantum sensing promising in near future applications using non-classical radiation fields and Gaussian state implementation, e.g., *quantum illumination (QI)*
- QI promises to outperform classical counterparts - even when
  - Low reflectivity
  - Low brightness
  - High thermal background

Classical radar

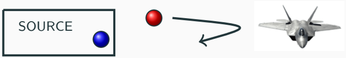
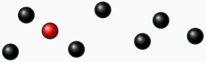
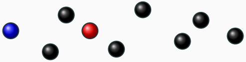


- Classical radar has not fundamentally changed in over 50 years. Obeys the *radar equation*:

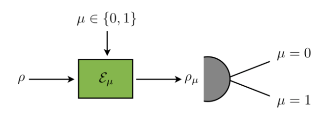
$$P_r = \frac{P_t G_t A_r \sigma F^4}{(4\pi)^2 R^4}, \qquad (1)$$

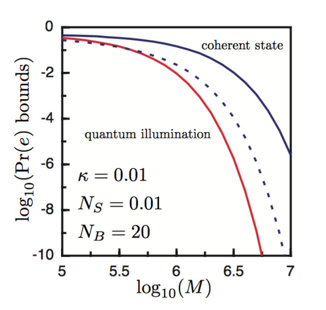
- Introducing noise $P_n = k_B T B_n F_n$, performance depends on SNR$= P_r/P_n$
- Vulnerable to a host of electronic countermeasures

Basic QI protocol

1	Entangle Signal and Idler mode pairs	
2	Send Signal, retain Idler	
3	Collect Return = Noise + Signal (?)	
4	Measure Idler with Return	
5	Detect:	(i) Correlations $\Rightarrow$ Target Present (ii) No correlations $\Rightarrow$ Target Absent

Quantum hypothesis testing





- Quantum radar is a task of binary QHT:
  - performance reduced to the distinguishablity of states ρ_μ.
- Requires $\rho_\mu^{\otimes M}$ with $M >> 1$

$$P_{\rm err}^{QI} = \frac{1}{2} \exp\left(-\frac{M\kappa N_S}{N_B}\right) \quad (2)$$

$$P_{\rm err}^{CS} = \frac{1}{2} \exp\left(-\frac{M\kappa N_S}{4N_B}\right) \quad (3)$$

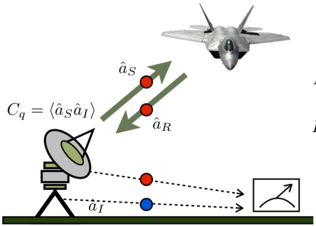
What we do...

- Generalise the definition of a quantum radar beyond QI:

  Any model that exploits a quantum part or device to outperform a corresponding classical radar under the same conditions of energy, range, etc.

- We progressively relax entanglement requirements of QI and study the corresponding detection performances to the point where the source becomes just-separable, i.e., a maximally-correlated separable state.
- At the same time, try to formulate equivalences between figures of merit between classical and quantum radar performance.

General quantum-correlated source



$$H_0 : \hat{a}_R = \hat{a}_B, \quad N_B >> 1$$

$$H_1 : \hat{a}_R = \sqrt{\kappa}\hat{a}_S + \sqrt{1-\kappa}\hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1-\kappa}$$

We consider a source modelled as a two-mode Gaussian state:

$$\mathbf{V}_{SI}^{gen} = \begin{pmatrix} S & C \\ C & S \end{pmatrix} \oplus \begin{pmatrix} S & -C \\ -C & S \end{pmatrix}, \tag{4}$$

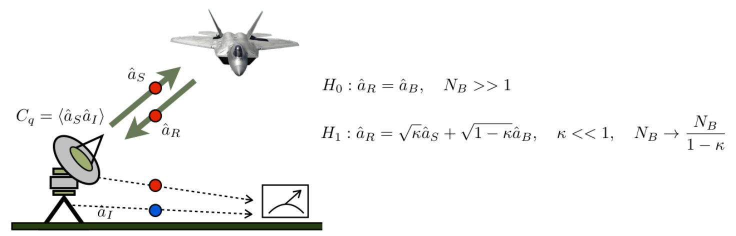
where

$$S := N_S + 1/2,$$

$$0 \leq C \leq \sqrt{S^2 - 1/4} = \sqrt{N_S(N_S+1)} = C_q,$$

$$C_d = N_S.$$

General quantum-correlated return



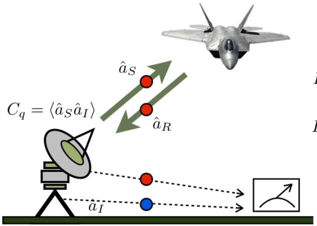
The joint state of our returning (R) mode and the retained idler is given by, under H_0 and H_1, respectively:

$$\mathbf{V}_{RI}^{(0)} = \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} \oplus \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix}, \tag{5}$$

$$\mathbf{V}_{RI}^{(1)} = \begin{pmatrix} A & \sqrt{\kappa}C \\ \sqrt{\kappa}C & S \end{pmatrix} \oplus \begin{pmatrix} A & -\sqrt{\kappa}C \\ -\sqrt{\kappa}C & S \end{pmatrix}, \tag{6}$$

where $B := N_B + 1/2$ and $A := \kappa N_S + B$.

Coherent state scenario - no access to idler!



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$$H_1 : \hat{a}_R = \sqrt{\kappa}\hat{a}_S + \sqrt{1-\kappa}\hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1-\kappa}$$

- Signal mode, annihilation operator $\hat{a}_S$, prepared in the coherent state $|\sqrt{N_S}\rangle$
- H_0: return is in a thermal state with mean photons per mode N_B, mean vector of zero and CM $(N_B + 1/2)\mathbf{1}_2$.
- H_1: return corresponds to a displaced thermal state with mean vector $(\sqrt{2\kappa N_S}, 0)$ and CM $(N_B + 1/2)\mathbf{1}_2$.

Hypothesis testing for quantum radar detection

- Detection probability $P_d := P(H_1|H_1)$.
- Two types of error may occur:
  type-I (false-alarm) $P_{fa} = P(H_1|H_0)$, and
  type-II (missed detection) $P_{md} = P(H_0|H_1)$.
- One may consider *asymmetric* testing in order to take in account discrepancies in error cost.
- A *symmetric* approach aims to obtain a global minimisation over all errors, irrespective of their origin. In this case, one considers the minimization of the average error probability

$$P_{\text{err}} := P(H_0)P(H_1|H_0) + P(H_1)P(H_0|H_1). \tag{7}$$

Symmetric hypothesis testing

- *Minimum* error probability is given by the Helstrom bound: $P_{\text{err}}^{\min} = [1 - D(\hat{\rho}_0, \hat{\rho}_1)]/2$, where $D(\hat{\rho}_0, \hat{\rho}_1) := |\hat{\rho}_0 - \hat{\rho}_1|/2$ is the trace distance.
- Analytically, use QCB,

$$P_{\text{err}}^{\min} \leq P_{\text{err}}^{\text{QCB}} := \frac{1}{2}\left(\inf_{0\leq s\leq 1} C_s\right), \quad C_s := \left(\hat{\rho}_0^s \hat{\rho}_1^{1-s}\right). \tag{8}$$

- Forgoing minimization, set $s = 1/2$ and define a simpler, though weaker, upper bound, the QBB

$$P_{\text{err}}^{\text{QBB}} := \frac{1}{2}\left(\sqrt{\hat{\rho}_0}\sqrt{\hat{\rho}_1}\right). \tag{9}$$

- For Gaussian states, closed analytical formulae exist for these!

Symmetric radar detection

- Assume typical condition of QI: $\kappa << 1$, $N_B >> 1$, $N_S << 1$.
- For a TMSV state, the minimum error probability satisfies

$$P_{\text{err}}^{\text{TMSV}} \leq e^{-M\kappa N_S/N_B}/2. \tag{10}$$

- Computed using the QBB, exponentially tight in limit of large M.
- Error-rate exponent has a factor of 4 advantage over the corresponding coherent-state transmitter

$$P_{\text{err}}^{\text{CS}} \leq e^{-M\kappa N_S/4N_B}/2. \tag{11}$$

Symmetric radar detection - generic source

- Extending Eq. (10) to the error probability for a generic source, we begin with $M = 1$ and assuming $\kappa << 1$, $N_B >> 1$ and $N_S << 1$.
- The QBB takes the form

$$P_{\text{err}}^{\text{gen}} \leq e^{-\kappa N_S g_C(N_S)/N_B}/2, \tag{12}$$

where $g_C(N_S) \propto C^2$, the amount of correlations existing between the signal and idler modes.

- Demanding equivalence of exponents in the TMSV limit $C \to C_q$, we find that the QBB for M probings becomes

$$P_{\text{err}}^{\text{gen}} \leq e^{-M\kappa N_S C^2/N_B C_q^2}/2. \tag{13}$$

Symmetric radar detection - generic source

- Comparing Eqs. (13) and (11), we see that a quantum-correlated transmitter beats the coherent state transmitter if $P_{\mathrm{err}}^{\mathrm{gen}} \leq P_{\mathrm{err}}^{\mathrm{CS}}$ which means

$$\frac{C^2}{C_q^2} \geq \frac{1}{4} \Rightarrow C \geq \frac{1}{2}\sqrt{N_S(N_S+1)}. \tag{14}$$

- Thus, the quadrature correlations required to outperform the semi-classical benchmark is half the value of those of a TMSV state.
- At the separable limit $C = N_S$ - the relation is only satisfied for $N_S \geq 1/3$, contradicting the assumption $N_S << 1$.
- A similar analysis holds if we relax the assumption of $N_S << 1$).
- The employment of a source at the separable limit is not capable of beating coherent states under symmetric testing.

Asymmetric hypothesis testing 1

- Consider M copies $\hat{\rho}_i^{\otimes M}$ of the state $\hat{\rho}_i$ encoding bit $i \in \{0, 1\}$.
- Binary outcome - two types of error, i.e., the type-I (false alarm) error

$$P_{\text{fa}} := P(H_1|H_0) = \left(E_1 \hat{\rho}_0^{\otimes M}\right), \tag{15}$$

and the type-II (missed detection) error

$$P_{\text{md}} := P(H_0|H_1) = \left(E_0 \hat{\rho}_1^{\otimes M}\right). \tag{16}$$

- These probabilities are dependent on the number M of copies and, for $M >> 1$, they both tend to zero, i.e.,

$$P_{\text{fa}} \simeq e^{-\alpha_R M}, \; P_{\text{md}} \simeq e^{-\beta_R M}, \tag{17}$$

where we define the 'error-exponents' or 'rate limits' as

$$\alpha_R = -\lim_{M \to +\infty} \frac{1}{M} \ln P_{\text{fa}}, \quad \beta_R = -\lim_{M \to +\infty} \frac{1}{M} \ln P_{\text{md}}. \tag{18}$$

Asymmetric hypothesis testing 2

- Place a relatively loose constraint $P_{\mathsf{fa}} < \epsilon$ on the type-I error, allowing us more freedom to minimize P_{md}.
- Quantum Stein's lemma: given this constraint, QRE is the optimal decay rate for the type-II error probability

$$D\left(\hat{\rho}_0||\hat{\rho}_1\right) = [\hat{\rho}_0(\ln\hat{\rho}_0 - \ln\hat{\rho}_1)] \tag{19}$$

- Tracking the type-II error exponent to second order (in M) depth, that is to order $\sqrt{M}$, allows one to define the QRE variance

$$V\left(\hat{\rho}_0||\hat{\rho}_1\right) = [\hat{\rho}_0(\ln\hat{\rho}_0 - \ln\hat{\rho}_1)^2] - [D\left(\hat{\rho}_0||\hat{\rho}_1\right)]^2. \tag{20}$$

- Optimal type-II error probability, for sample size M:

$$P_{\mathsf{md}} = \exp\left\{-\left[MD\left(\hat{\rho}_0||\hat{\rho}_1\right) + \sqrt{MV\left(\hat{\rho}_0||\hat{\rho}_1\right)}\Phi^{-1}(\epsilon) + \mathcal{O}(\log M)\right]\right\}, \tag{21}$$

where $\epsilon \in (0,1)$ bounds P_{fa} and $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} dx \exp\left(-x^2/2\right)$.

Asymmetric radar detection

- We evaluate the QRE and QRE-variance to first order in N_B by taking an asymptotic expansion for large N_B while keeping N_S fixed. We obtain

$$D_{\text{gen}} := D\left(\hat{\rho}_{RI}^{(0)}||\hat{\rho}_{RI}^{(1)}\right) = \frac{\kappa C^2}{N_B}\ln\left(1+\frac{1}{N_S}\right) + \mathcal{O}\left(N_B^{-2}\right), \qquad (22)$$

$$V_{\text{gen}} := V\left(\hat{\rho}_{RI}^{(0)}||\hat{\rho}_{RI}^{(1)}\right) = \frac{\kappa C^2(2N_S+1)}{N_B}\ln^2\left(1+\frac{1}{N_S}\right) + \mathcal{O}\left(N_B^{-2}\right). \qquad (23)$$

- For coherent states these quantities take the form

$$D_{\text{CS}} := D\left(\hat{\rho}_{\text{CS}}^{(0)}||\hat{\rho}_{\text{CS}}^{(1)}\right) = \kappa N_S \ln\left(1+\frac{1}{N_B}\right) \simeq \gamma + \mathcal{O}\left(N_B^{-2}\right), \qquad (24)$$

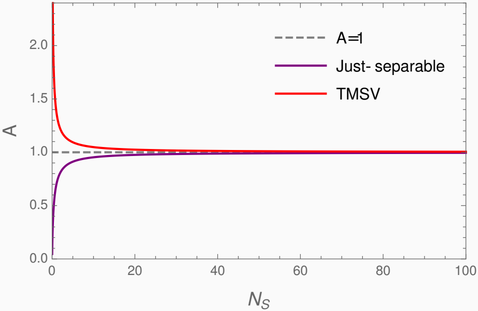
$$V_{\text{CS}} := V\left(\hat{\rho}_{\text{CS}}^{(0)}||\hat{\rho}_{\text{CS}}^{(1)}\right) = \kappa N_S(2N_B+1)\ln^2\left(1+\frac{1}{N_B}\right) \simeq 2\gamma + \mathcal{O}\left(N_B^{-2}\right), \qquad (25)$$

- Signal-to-noise ratio (SNR): $\gamma := \frac{\kappa N_S}{N_B}$, usually expressed in decibels (dB) via $\gamma_{\text{dB}} = 10\log_{10}\gamma$.

Asymmetric performance comparison

Define an error-exponent advantage over coherent states:

$$A(C, N_S) := \frac{D_{\text{gen}}}{D_{\text{CS}}} = \frac{C^2}{N_S} \ln\left(1 + \frac{1}{N_S}\right). \tag{26}$$



- Benefits of max-entanglement for QI only for very small energies.
- For increasing N_S, the ratio $A \to 1$, irrespective of source specification.
- Just-separable source quickly approaches the performance of QI already at about 20 photons.

Receiver operating characteristic

Study mis-detection probability vs false-alarm probability for generic Gaussian source and coherent state classical benchmark.

- Optimal: phase is maintained - homodyne + coherent integration
- Deterministic phase shift - use heterodyne + coherent integration.
- Otherwise: unknown phase shift - heterodyne + non-coherent integration, given by Marcum's Q-function
- Marcum's Q -function may be overestimated by assuming single coherent pulse with MN_S photons.

ROC: generic quantum source

- The ROC $P_{\mathrm{md}} = P_{\mathrm{md}}(P_{\mathrm{fa}})$ of the gen. quantum source can be upper bounded:

$$P_{\mathrm{md}} \leq \tilde{P}_{\mathrm{md}}^{\mathrm{gen}} = \exp\left\{-\left[\sqrt{\frac{M\gamma}{N_S}}\Lambda C \ln\left(1+\frac{1}{N_S}\right) + \mathcal{O}(N_B^{-1}, 1)\right]\right\}, \tag{27}$$

$$\Lambda := \left(\sqrt{\frac{M\gamma}{N_S}} C + \sqrt{2N_S + 1}\Phi^{-1}(P_{\mathrm{fa}})\right). \tag{28}$$

- Sufficiently large M ($\gtrsim 10^7$) and large N_B ($\gtrsim 10^2$).

ROC: coherent states

- Optimal - homodyne + coherent integration and binary testing:

$$P^{\text{fa}}_{\mathrm{CS},\text{hom}}(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{M(2N_B+1)}}\right), \tag{29}$$

$$P^{\text{md}}_{\mathrm{CS},\text{hom}}(x) = \frac{1}{2}\,\text{erfc}\left(\frac{M\sqrt{2\kappa N_S} - x}{\sqrt{M(2N_B+1)}}\right), \tag{30}$$

where $\text{erfc}(z) := 1 - 2\pi^{-1/2}\int_0^z \exp(-t^2)dt$ is the complementary error function.

- Lower bound - assume single coherent state with mean number of photons MN_S so that the total SNR is given by $M\gamma$:

$$P^{\text{Marcum}}_{\text{md}} = 1 - Q\left(\sqrt{2M\gamma}, \sqrt{-2\ln P_{\text{fa}}}\right), \tag{31}$$

where the Marcum Q-function is defined as

$$Q(x,y) := \int_y^\infty dt\ te^{-(t^2+x^2)/2} I_0(tx). \tag{32}$$

Parameters

Focusing on short-range ($\sim$ 1m) applications, e.g. security or biomedical:

- For $\nu = 1$GHz (L band) and $T = 290\mathrm{K}$ (room temperature), we get $N_B \simeq 6 \times 10^3$ photons (bright noise)
- Assume broadband pulses, with 10% bandwidth (100MHz), so that their individual duration $\sim$ 10ns. Using $M = 10^8$ pulses then we have an integration time $\sim$ 1s - acceptable for slow-moving/still objects.
- Low-energy applications - assume $N_S = 1$ mean photon per pulse.
- What about the SNR γ? - Related to overall reflectivity κ, estimated by the radar equation.

Parameters

- The radar equation relates returning signal power P_R to the transmitted signal power P_T:

$$P_R = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4} P_T \tag{33}$$

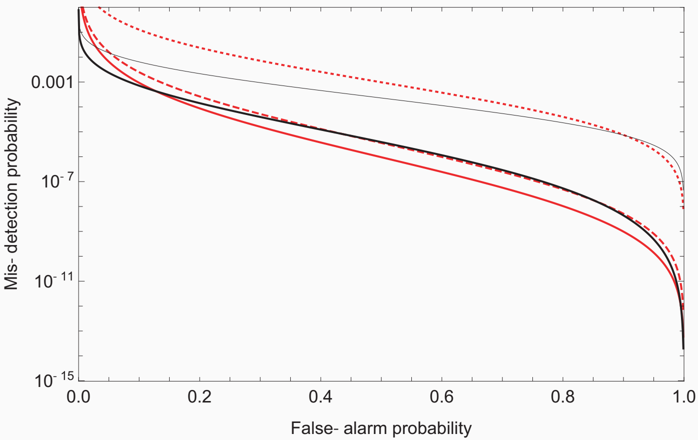
- At the same time, we can also write κ as the ratio:

$$\kappa = \frac{P_R}{P_T} = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4}, \tag{34}$$

- Assume $F = 1$ (no free-space loss) and ideal pencil beam so that solid angle δ is exactly subtended by the target's σ (valid at short range) so that $G = 4\pi/\delta = 4\pi R^2/\sigma$.
- Then,

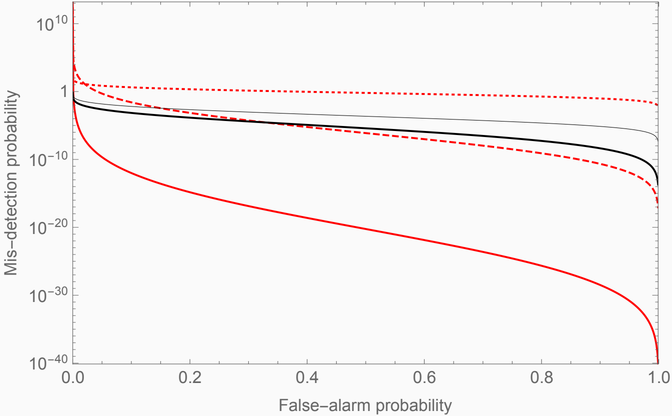
$$\kappa = \frac{A_R}{(4\pi R)^2}, \quad R = \frac{1}{4\pi}\sqrt{\frac{A_R}{\kappa}}. \tag{35}$$

ROC comparison: $N_S = 1, R = 1m$



- Red curves: Gaussian state QI with $C(p) = pC_d + (1-p)C_q$.
  Just-separable, $p = 0$, (dotted), maximal entanglement ($p = 1$), solid, and intermediate correlations ($p = 1/6$), dashed.
- Black curves: Classical coherent state benchmark
  Optimal homodyne detection, thick, and lower bound, thin.

ROC comparison: $N_S = 0.01, R = 0.1m$



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  Just-separable, $p = 0$, (dotted), maximal entanglement ($p = 1$), solid, and intermediate correlations ($p = 1/2$), dashed.
- Black curves: Classical coherent state benchmark
  Optimal homodyne detection, thick, and lower bound, thin.

Concluding remarks

- We have investigated how to loosen QI transmitter requirements.
- Scenarios of symmetric and asymmetric testing where we test the quantum performance with respect to suitable classical benchmarks.
- Quantum advantage still exists by using Gaussian sources which are not necessarily maximally entangled.
- Short ranges only!
  (so spherical beam spreading does not involve too many dBs of loss, a major killing factor for any quantum radar design based on the exploitation of quantum correlations)
- A short-range, low-power radar is potentially interesting as noninvasive scanning tool for biomedical applications but also for security and safety purposes.

Current and ongoing work

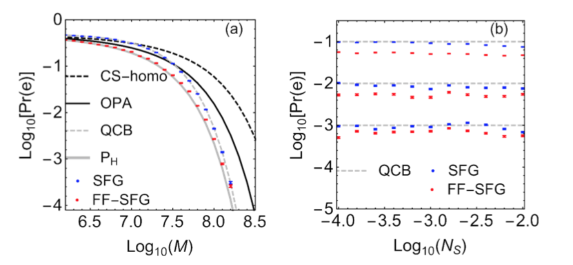
- Previous results assumes use of *optimal* receivers achieving

$$R_{QI} = \frac{\kappa N_S}{N_B} \qquad \text{vs.} \qquad R_{CS} = \frac{\kappa N_S}{4N_B} \tag{36}$$

- Practical receiver designs include Guha Erkman (2009) - Optical Parametric Amplifer (OPA) and Phase Conjugating (PC) receiver achieving

$$R_{PC/OPA} \simeq \frac{\kappa N_S}{2N_B}, \qquad N_S \ll 1, \kappa \ll 1, N_B \gg 1 \tag{37}$$

and Zhuang (2016) FF-SFG receiver (non-linear)



Noisy receivers: set up

Source:

$$\mathbf{V}_{S,I} = \frac{1}{2}\begin{pmatrix} \nu\mathbf{1} & c\mathbf{Z} \\ c\mathbf{Z} & \mu\mathbf{1} \end{pmatrix}, \quad \begin{cases} \mathbf{1} := \mathrm{diag}(1,1), \\ \mathbf{Z} := \mathrm{diag}(1,-1), \end{cases} \tag{38}$$

where $\nu := 2N_S + 1$, $\mu := 2N_I + 1$ and c quantifies the quadrature correlations between the two modes such that $0 \leq c \leq 2\sqrt{N_S(N_I+1)}$.

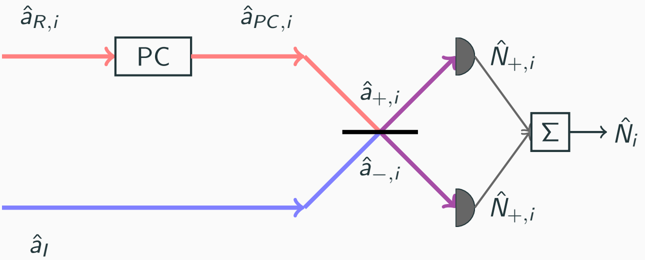
Return:

$$\mathbf{V}^0_{R,I} = \frac{1}{2}\begin{pmatrix} \omega\mathbf{1} & 0 \\ 0 & \mu\mathbf{1} \end{pmatrix}, \tag{39}$$

$$\mathbf{V}^1_{R,I} = \frac{1}{2}\begin{pmatrix} \gamma\mathbf{1} & \sqrt{\kappa}c\mathbf{Z} \\ \sqrt{\kappa}c\mathbf{Z} & \mu\mathbf{1} \end{pmatrix}, \tag{40}$$

where we set $\omega := 2N_B + 1$ and $\gamma := 2\kappa N_S + \omega$.

Noisy receivers: PC



SNR for generic source, c, is given by

$$\mathrm{SNR}_{\mathrm{PC}} = \frac{\kappa c^2}{\left(\sqrt{\kappa c^2 + \mu(1+\gamma)} + \sqrt{\mu(1+\omega)}\right)^2}. \quad (41)$$

This directly relates to its error probability after M uses, for equally-likely hypotheses, satisfying

$$P_{\mathrm{PC}}^{(M)} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{M\mathrm{SNR}_{\mathrm{PC}}}\right). \quad (42)$$

Noisy receiver performance

Using our SNR, write

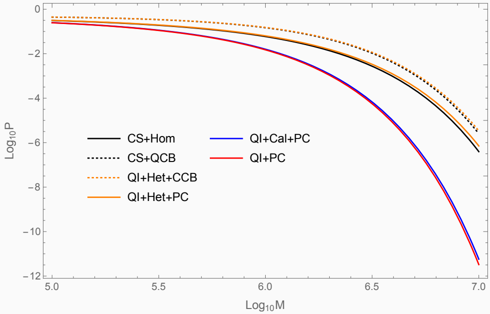
$$\mu \to \mu' = \mu + \varepsilon_I, \tag{43}$$

and

$$\begin{aligned} \omega \to \omega' = \omega + \varepsilon_R, \text{ under } H_0, \\ \gamma \to \gamma' = \gamma + \varepsilon_R, \text{ under } H_1. \end{aligned} \tag{44}$$

Assume $\varepsilon_{I(R)} = 1$ (heterodyne).
QI+PC: Entangled TMSV source with PC receiver
QI+Het+PC: $\varepsilon_I = \varepsilon_R = 1$ before the PC receiver
QI+Cal+PC: $\varepsilon_R = 1$ and $\varepsilon_I = 0$

Noisy receiver performance



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QI+PC: Entangled TMSV source with PC receiver
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$$\mathrm{SNR}_{\mathrm{QI+Cal+PC}} \to \mathrm{SNR}_{\mathrm{QI+PC}} = \frac{(1+N_I)\kappa N_S}{2N_B(1+2N_I)}, \tag{45}$$

and

$$\mathrm{SNR}_{\mathrm{QI+Het+PC}} \to \mathrm{SNR}_{\mathrm{CS+Hom}} = \frac{\kappa N_S}{4N_B}. \tag{46}$$

The maximal advantage of QI+PC over CS+Hom is given by

$$\frac{\mathrm{SNR}_{\mathrm{QI+PC}}}{\mathrm{SNR}_{\mathrm{CS+Hom}}} = \frac{2(1+N_I)}{1+2N_I} \to 2 \text{ for } N_I \ll 1. \tag{47}$$

Thanks for listening!